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Pseudofunctorial behavior of Cousin complexes on formal schemes

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Abstract. On a suitable category of formal schemes equipped with codimension functions we construct a *canonical pseudofunctor* $(-)^\sharp$ taking values in the corresponding categories of *Cousin complexes.* Cousin complexes on such a formal scheme $\mathcal{X}$ functorially represent derived-category objects $\mathcal{F}$ by the local cohomologies $H_x^{\mathrm{codim}(x)}\mathcal{F}$ $(x \in \mathcal{X})$ together with "residue maps" from the cohomology at x to that at each immediate specialization of x; this representation is faithful when restricted to $\mathcal{F}$ which are *Cohen-Macaulay* (CM), i.e., $H_x^i\mathcal{F} = 0$ whenever $i \neq \mathrm{codim}(x)$. Formal schemes provide a framework for treating local and global duality as aspects of a single theory. One motivation has been to gain a better understanding of the close relation between local properties of residues and global variance properties of dualizing complexes (which are CM). Our construction, depending heavily on local phenomena, is inspired by, but generalizes and makes more concrete, that of the classical pseudofunctor $(-)^\Delta$ taking values in residual complexes, on which the proof of Grothendieck's (global) Duality Theorem in Hartshorne's "Residues and Duality" is based. Indeed, it is shown in the following paper by Sastry that $(-)^\sharp$ is a good "concrete approximation" to the fundamental duality pseudofunctor $(-)^!$. The pseudofunctor $(-)^\sharp$ takes residual complexes to residual complexes, so contains a canonical representative of $(-)^\Delta$; and it generalizes as well several other functorial (but not pseudofunctorial) constructions of residual complexes which appeared in the 1990s.

Contents

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1. Introduction and main results

1.1. Introduction. At the heart of the foundations of Grothendieck Duality lies the duality pseudofunctor $(-)^!$ described in the Preface. As indicated in §0.6, under suitable hypotheses on the map $f\colon \mathscr{X} \to \mathscr{Y}$ of noetherian formal schemes, the functor $f^!\colon \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})$ can be realized in terms of *dualizing complexes* on $\mathscr{X}$ and $\mathscr{Y}$. Anyway, the study of dualizing complexes has its own importance, for example as a natural generalization of the oft-appearing notion of *dualizing sheaf.* Concrete models for dualizing complexes—the residual complexes—are found in the category of Cousin complexes. This category, which, among other virtues, is an abelian subcategory of the usual category of complexes, provides fertile ground for a concrete pseudofunctorial (or "variance") theory modeled after that of $(-)^!$.

Our purpose here is to develop such a canonical pseudofunctorial construction of Cousin complexes over a suitably general category of formal schemes. (The notion of "pseudofunctor" is recalled at the beginning of §4.) The construction is motivated by well-known concrete realizations of the duality pseudofunctor $(-)^!$, and indeed, is shown in [**24**] to provide a "concrete approximation" to $(-)^!$.

Before stating the main result (in §1.3), we highlight some of its salient features, and relate it to some results in the literature on Cousin complexes.

First, the underlying category $\mathbb{F}$ on which we work is that of morphisms of noetherian formal schemes with additional mild hypotheses specified below in §1.2. In particular $\mathbb{F}$ contains many ordinary schemes, which can be regarded as formal schemes whose structure sheaf (of topological rings) has the discrete topology. Also, $\mathbb{F}$ contains the opposite category of the category $\mathfrak{C}$ of those local homomorphisms of complete noetherian local rings which induce a finitely generated extension at the residue fields. A key advantage of working in a category of formal schemes is that it offers a framework for treating local and global duality as aspects of a single theory, see, e.g., [**2**, §2]. This paper continues efforts to generalize all of Grothendieck's duality theory to the context of formal schemes.

Second, while originally inspired by a study of the classical construction in [**11**, Chap. 6] of a pseudofunctor on residual complexes over schemes (see also [**5**, Chap. 3]), we work more generally with Cousin complexes, the only restriction being that the underlying modules be quasi-coherent and torsion. (Over ordinary schemes the "torsion" condition is vacuous.) The pseudofunctor we construct does however take residual complexes to residual complexes (Proposition 9.2.2). So our construction generalizes the one in [**11**].

Our construction is based on the canonical pseudofunctor of Huang ([**13**]), which is defined over $\mathfrak{C}$. This pseudofunctor expands readily to one with values in the category of graded objects underlying Cousin complexes, that is, Cousin complexes with vanishing differentials. Most of our effort lies in working out what to do with nontrivial differentials.

Many details are thus already absorbed into the local theory of residues, through its basic role in Huang's work. In fact much of what we do in this paper comes down ultimately to the relation between local operations involving residues and global operations on Cousin complexes.

Finally, we note that several canonical constructions of residual complexes came out during the 1990s, see [**14**], [**15**], [**27**], [**23**] (some of which also use [**13**]). These constructions—functorial, but not pseudofunctorial—lead by various methods to special cases of our result.

1.2. Terminology and remarks on basic issues. The main theorem of this paper is stated in §1.3. To prepare the way, we first describe various underlying notions, referring to later sections for precise definitions; and give a preliminary discussion of some of the basic issues involved.

Consider the category $\mathbb{F}$ whose objects are all (in some universe) the *noetherian universally catenary formal schemes admitting a codimension function* and whose morphisms $\mathcal{X}' \to \mathcal{X}$ are all those formal-scheme maps which are *essentially of pseudo-finite type* (§2.1). We will usually work with the refined category $\mathbb{F}_c$ whose objects are pairs $(\mathcal{X}, \Delta)$ with $\mathcal{X}$ in $\mathbb{F}$ and Δ a codimension function on $\mathcal{X}$, and whose morphisms $(\mathcal{X}', \Delta') \to (\mathcal{X}, \Delta)$ are those $\mathbb{F}$-maps $f\colon \mathcal{X}' \to \mathcal{X}$ such that for any $x' \in \mathcal{X}'$ and $x := f(x')$, $\Delta(x) - \Delta'(x')$ is the transcendence degree of the residue field extension $k(x')/k(x)$ ($k(x)$ being the residue field of the local ring $\mathcal{O}_{\mathcal{X},x}\dots$, see 2.1.2).

Let $(\mathcal{X}, \Delta) \in \mathbb{F}_c$. A *Cousin complex on* $(\mathcal{X}, \Delta)$, or a Δ*-Cousin complex on* $\mathcal{X}$, is an $\mathcal{O}_\mathcal{X}$-complex $\mathcal{M}^\bullet$ such that for each $n \in \mathbb{Z}$, $\mathcal{M}^n$ is the direct sum of a family of $\mathcal{O}_\mathcal{X}$-submodules $(i_x M_x)_{x \in \mathcal{X},\, \Delta(x)=n}$, where M_x is an $\mathcal{O}_{\mathcal{X},x}$-module and $i_x M_x$ is the extension by 0 of the constant sheaf $\overline{M}_x$ on the closure $\overline{\{x\}}$ such that for all nonempty open subsets V of $\overline{\{x\}}$, $\overline{M}_x(V)$ is the $\mathcal{O}_{\overline{\{x\}}}(V)$-module M_x (see §3.2). For such an $\mathcal{M}^\bullet$, and any $x \in \mathcal{X}$, $i_x M_x$ is uniquely determined: $i_x M_x = \varGamma_{\overline{\{x\}}}\mathcal{M}^{\Delta(x)}$, where $\varGamma_{\overline{\{x\}}}$ is the subfunctor of the identity functor taking any $\mathcal{O}_\mathcal{X}$-module to its sheaf of sections supported in $\overline{\{x\}}$. Thus M_x, which is $(i_x M_x)(U)$ for any open neighborhood U of x, is determined by $\mathcal{M}^\bullet$ and x, and so we denote it by $\mathcal{M}^\bullet(x)$.

Let $\mathrm{Coz}_\Delta(\mathcal{X})$ be the full subcategory of the category of $\mathcal{O}_\mathcal{X}$-complexes with objects those Δ-Cousin $\mathcal{O}_\mathcal{X}$-complexes $\mathcal{M}^\bullet$ whose underlying graded modules are *quasi-coherent torsion* $\mathcal{O}_\mathcal{X}$*-modules* (§2.2, §2.3). By 2.3.5 this last condition on $\mathcal{M}^\bullet$ means simply that for any $x \in \mathcal{X}$, each element of $\mathcal{M}^\bullet(x)$ is annihilated by some power of the maximal ideal m_x of $\mathcal{O}_{\mathcal{X},x}$, or, as we will say, $\mathcal{M}^\bullet(x)$ is a *zero-dimensional* $\mathcal{O}_{\mathcal{X},x}$*-module*. Thus we can, and will, view $\mathcal{M}^\bullet(x)$ as a zero-dimensional module over the m_x-adic completion $\widehat{\mathcal{O}_{\mathcal{X},x}}$. The category of such modules will be denoted by $(\mathcal{O}_x)_\#$.

Let $\mathrm{Coz}^0_\Delta(\mathcal{X})$ be the full subcategory of $\mathrm{Coz}_\Delta(\mathcal{X})$ with objects those complexes whose differentials are all zero. For any $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ the natural map

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, i_x M_x) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x, M_x)$$

is easily seen to be bijective (i.e., i_x is right-adjoint to the functor $\mathcal{F} \mapsto \mathcal{F}_x$). Hence

$$[\,\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(i_y M_y, i_x M_x) \neq 0\,] \implies [\,(i_y M_y)_x \neq 0\,] \implies [\,x \in \overline{\{y\}}\,].$$

Consequently, a morphism $\varphi\colon \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ in $\mathrm{Coz}^0_\Delta(\mathcal{X})$ is the same thing as a family of $\widehat{\mathcal{O}_{\mathcal{X},x}}$-homomorphisms $\big(\varphi(x)\colon \mathcal{M}^\bullet(x) \to \mathcal{N}^\bullet(x)\big)_{x\in\mathcal{X}}$. It follows that $\mathrm{Coz}^0_\Delta(\mathcal{X})$ is naturally equivalent to the disjoint union of the family of categories $\big((\mathcal{O}_x)_\#\big)_{x\in\mathcal{X}}$. (It also follows that the category $\mathrm{Coz}_\Delta(\mathcal{X})$ is *abelian*.)

The *forgetful functor* $\mathtt{Fgt}_\Delta(\mathcal{X})\colon \mathrm{Coz}_\Delta(\mathcal{X}) \to \mathrm{Coz}^0_\Delta(\mathcal{X})$ sends a complex to its underlying graded module, and a map of complexes to itself.

Example 1.2.1. The scheme $\mathcal{X} = \mathrm{Spec}(\mathbb{Z})$ is an affine formal scheme with (0) as an ideal of definition. Every $\mathcal{O}_\mathcal{X}$-module is a torsion module. Let Δ be the codimension function sending the generic point to 0 and all other points to 1.

The natural surjection $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \cong \oplus_p\,(\mathbb{Q}/\mathbb{Z}_{p\mathbb{Z}})$ (with p ranging over the positive primes, so that $p\mathbb{Z}$ ranges over all nonzero prime ideals) may be viewed as a $\mathbb{Z}$-complex

concentrated in degrees 0 and 1. Application of the sheafification functor $^\sim$ produces a complex in $\mathrm{Coz}_\Delta(\mathcal{X})$

$$\mathbb{Q}^\sim \longrightarrow \bigoplus_p (\mathbb{Q}/\mathbb{Z}_{p\mathbb{Z}})^\sim.$$

This is a residual complex, with homology $\mathcal{O}_\mathcal{X}$ concentrated in degree 0.

Analogous statements hold for any Dedekind domain and its fraction field. For a generalization to any irreducible regular scheme, with the codimension function which sends the generic point to 0, see [**11**, p. 304, Example].

By a Coz-valued (resp. Coz^0-valued) pseudofunctor on $\mathbb{F}_\mathrm{c}$ we shall always mean a *contravariant pseudofunctor* (see §4) on the category $\mathbb{F}_\mathrm{c}$ which assigns to any $(\mathcal{X},\Delta)$ in $\mathbb{F}_\mathrm{c}$ the category $\mathrm{Coz}_\Delta(\mathcal{X})$ (resp. $\mathrm{Coz}^0_\Delta(\mathcal{X})$).

We aim to construct a canonical Coz-valued pseudofunctor on $\mathbb{F}_\mathrm{c}$, one which, when restricted to pseudo-proper maps will be shown in [**24**] to be right adjoint to the direct image functor of Cousin complexes, and further, to be a "concrete approximation" to the basic duality pseudofunctor $(-)^!$. Let us discuss briefly some of the issues involved.

To begin with, the following examples serve as inspiration for our construction. These are generalized Cousin versions of well-known concrete realizations of $(-)^!$ on ordinary schemes.

Example 1.2.2. Let $f\colon (\mathcal{X},\Delta_1) \to (\mathcal{Y},\Delta)$ be an $\mathbb{F}_\mathrm{c}$-map and $\mathcal{M}^\bullet \in \mathrm{Coz}_\Delta(\mathcal{Y})$.

(i) If f is smooth, with relative dimension d (a locally constant function, see 2.4.2, 2.6.2), then the complex $E_{\Delta_1}\mathbf{R}\mathit{\Gamma}'_\mathcal{X}(f^*\mathcal{M}^\bullet \otimes_\mathcal{X} \omega_f[d])$ is in $\mathrm{Coz}_{\Delta_1}(\mathcal{X})$ (see 2.6.4 and §5.1), where:

—E_{Δ_1} is the Cousin functor corresponding to the filtration of $\mathcal{X}$ induced by Δ_1 (see §3.2);

—$\mathit{\Gamma}'_\mathcal{X}$ is the subfunctor of the identity functor on $\mathcal{O}_\mathcal{X}$-modules associating to any such module the largest submodule each section of which, over any open set $\mathcal{U}$, is annihilated by some open $\mathcal{O}_\mathcal{U}$-ideal (see §2.3), and $\mathbf{R}\mathit{\Gamma}'_\mathcal{X}$ is the right-derived functor of $\mathit{\Gamma}'_\mathcal{X}$;

—ω_f is the d-th exterior power of the complete differential module of $\mathcal{X}$ over $\mathcal{Y}$ (see §2.5, 2.6.1, 2.6.4); and $[d]$ is the usual translation operator on complexes, where both instances of "d" should be replaced separately on each connected component of $\mathcal{X}$ by the value of d on that component.

Thus we get a functor $\mathbb{E}_f\colon \mathrm{Coz}_\Delta(\mathcal{Y}) \to \mathrm{Coz}_{\Delta_1}(\mathcal{X})$. In §5 we expand this functor to a pseudofunctor on smooth maps. This expansion is canonical in that it involves no choices other than the sign convention fixed in §1.4(v) to handle the relation between $\otimes$ and translation of complexes. For open immersions f, $\mathbb{E}_f$ is then canonically pseudofunctorially isomorphic to the usual restriction functor f^*.

(ii) If f is a closed immersion ([**10**, p. 442]), the complex $f^{-1}\mathcal{H}om_\mathcal{Y}(f_*\mathcal{O}_\mathcal{X}, \mathcal{M}^\bullet)$ is in $\mathrm{Coz}_{\Delta_1}(\mathcal{X})$ (see 6.1.1).

Thus we get a functor $f^\flat\colon \mathrm{Coz}_\Delta(\mathcal{Y}) \to \mathrm{Coz}_{\Delta_1}(\mathcal{X})$. In §6 we canonically expand this functor to a pseudofunctor on closed immersions. For open-and-closed immersions f, $f^\flat$ is canonically pseudofunctorially isomorphic to the usual restriction functor f^*.

What we want is to glue these examples together, i.e., to get a Coz-valued pseudofunctor $(-)^\sharp$ on $\mathbb{F}_c$ whose restriction to smooth maps (resp. closed immersions) is isomorphic to $\mathbb{E}_{(-)}$ (resp. $(-)^\flat$). The result should be *canonical,* i.e., unique up to unique isomorphism, as explicated in the Main Theorem in §1.3.

For this we need concrete descriptions of a number of "glueing" isomorphisms involving $\mathbb{E}_{(-)}$ and $(-)^\flat$, associated to various compositions of smooth maps and closed immersions. For instance, suppose f and $\mathcal{M}^\bullet$ are as in 1.2.2, and assume further that f admits factorizations $f = h_1 i_1$ and $f = h_2 i_2$ where h_1, h_2 are smooth maps in $\mathbb{F}_c$ and i_1, i_2 are closed immersions. (A map g will be called *factorizable* if $g = hi$ with h a smooth map in $\mathbb{F}_c$ and i a closed immersion.) Then, at the least, there has to be a canonical functorial isomorphism $i_1^\flat \mathbb{E}_{h_1} \xrightarrow{\sim} i_2^\flat \mathbb{E}_{h_2}$.

Moreover, for a general map g in $\mathbb{F}_c$, one that is not factorizable, it is not immediately obvious how to define $g^\sharp$, canonically or otherwise. This issue is related to the previous one in that if one somehow obtains a good definition of $f^\sharp$ for any factorizable f, then for general g one could use *local* factorizations (2.4.4) to canonically define $(g\big|_{\mathcal{U}_\lambda})^\sharp$ for a suitable open cover $\{\mathcal{U}_\lambda\}$ of $\mathcal{X}$, and, having noted that $\mathbb{E}_f$ for f an open immersion "is" the restriction functor f^*, anticipate further that canonicity would enable pasting the various $(g\big|_{\mathcal{U}_\lambda})^\sharp$ together to form a global $g^\sharp$.

Thus the issue of canonicity is basic. One must specify with care all the maps that go into the construction, and relations between them, and in particular, pay close attention to signs.

We now briefly describe our construction of $(-)^\sharp$. One can think of a Cousin complex as comprising two parts, its underlying graded object and its differential; and accordingly the problem of canonicity is addressed in two stages. We start with a pseudofunctor $(-)^\natural$ on graded objects only—in other words, $(-)^\natural$ is a Coz^0-valued pseudofunctor on $\mathbb{F}_c$. The idea is to use Huang's work ([**13**, Chap. 6]) wherein he constructs a *canonical* pseudofunctor $(-)_\#$ which takes values in categories $R_\#$ of zero-dimensional modules over local rings $(R, \mathfrak{m})$ in the previously-mentioned category $\mathfrak{C}$, that is, modules such that each element is annihilated by some power of the maximal ideal $\mathfrak{m}$. A map $f\colon \mathcal{X} \to \mathcal{Y}$ such as in 1.2.2 provides, for each $x \in \mathcal{X}$, a map $\widehat{f_x}$ in $\mathfrak{C}$, namely, the naturally induced map $\widehat{\mathcal{O}_{\mathcal{Y}, f(x)}} \to \widehat{\mathcal{O}_{\mathcal{X},x}}$. We can therefore define $f^\natural$ by using Huang's $(-)_\#$, pointwise; and furthermore define $(-)^\natural$ as a pseudofunctor in a similar manner. (Actually, we will need to impose a subsequently important sign modification on $(-)_\#$, see proof of 4.3.1.)

Then we upgrade the pseudofunctor $(-)^\natural$ to a pseudofunctor $(-)^\sharp$ at the level of complexes. This means setting up a canonical differential on every $f^\natural\mathcal{M}^\bullet$. For that purpose we show that in each case, (i) or (ii), of 1.2.2, the complex obtained in $\mathrm{Coz}_{\Delta_1}(\mathcal{X})$ is, at the graded level, canonically isomorphic to $f^\natural\mathcal{M}^\bullet$. (See §§8.1 and 8.2.) So we can transfer the differential from $\mathbb{E}_f\mathcal{M}^\bullet$ (resp. $f^\flat$) to $f^\natural\mathcal{M}^\bullet$ when f is smooth (resp. a closed immersion). The important step, the one lying at the heart of this paper, is to show that *if f is a factorizable map, then the differentials for $f^\natural\mathcal{M}^\bullet$ so obtained via different factorizations of f are the same.* This is accomplished in Proposition 8.3.3. So we have a canonical differential for $f^\natural\mathcal{M}^\bullet$ whenever f is factorizable. As mentioned before, a definition of the differential in the general case, indeed a canonical one, then follows easily, see §8.4. Upgrading of the remaining data of $(-)^\natural$ is straightforward.

We can summarize as follows. Let us say that a Coz or Coz^0 functor is *canonical* if it satisfies the appropriate analogs of the conditions on pages 319–321 of [**11**], with "finite morphism" replaced by "closed immersion". (Some of these conditions spell out what we have said above about pasting the basic examples together; and the others say that this pasting should be compatible with certain base-change and residue isomorphisms, as explained in detail in §§6 and 7 below.) In addition, the pseudofunctor should commute with restriction to open subsets. Any two canonical pseudofunctors are canonically isomorphic. What is accomplished in §§5-7 is, in essence, showing that *the* Coz^0*-valued pseudofunctor derived as above from Huang's zero-dimensional pseudofunctor is, modulo some sign modifications, canonical;* and what is done in §8 shows, in essence, that *any canonical* Coz^0*-valued pseudofunctor can be upgraded to a canonical* Coz*-valued pseudofunctor.*

1.3. Outline of construction of $(-)^\sharp$**.** In more detail, our construction of $(-)^\sharp$ is realized through the steps (A)-(D) below. The main theorem is stated after (C). Step (D) occupies the bulk of this paper.

(A) We start with the punctual case, i.e., we (temporarily) restrict attention to the full subcategory $\mathbb{F}_c^\circ \subset \mathbb{F}_c$ with objects those $(\mathcal{X}, \Delta)$ such that the underlying space of $\mathcal{X}$ is a single point—so that Δ can be identified with a single integer. Associating to each such $(\mathcal{X}, \Delta)$ the complete local ring $A_{\mathcal{X}} := \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ leads to an antiequivalence from $\mathbb{F}_c^\circ$ to the category of pairs (A, n) with $A \in \mathfrak{C}$ (see §1.1) and $n \in \mathbb{Z}$, a morphism $(A, n) \to (B, m)$ being a $\mathfrak{C}$-map $A \to B$ of residual transcendence degree $n - m$. The category $\mathrm{Coz}_\Delta(\mathcal{X})$ is isomorphic to the category of zero-dimensional $A_{\mathcal{X}}$-modules. In effect, then, the punctual case is covered by [**13**], wherein Huang constructs a *canonical* covariant pseudofunctor $(-)_\#$ on $\mathfrak{C}$ taking values in zero-dimensional modules.

For our purposes a variant $(-)_\sharp$ of Huang's pseudofunctor $(-)_\#$ is more convenient. The changes made are just to multiply the comparison isomorphisms of $(-)_\#$ by a suitable $\pm$ factor, and to reverse the order in which modules appear in certain tensor products. The description of $(-)_\sharp$ begins with the following data, cf. Example 1.2.2:

(i) For any formally smooth $\mathfrak{C}$-map $\phi\colon R \to S$ with $r :=$ dimension of $S/\mathfrak{m}_R S$ and $t :=$ residual transcendence degree, and $\omega_{S/R} := \widehat{\Omega^{r+t}_{S/R}}$, the $(r+t)$-th exterior power of the complete module of R-differentials of S, and for any zero-dimensional R-module M, a specific isomorphism

$$\phi_\sharp M \xrightarrow{\ \sim\ } H^r_{\mathfrak{m}_S}(M \otimes_R \omega_{S/R}).$$

(ii) For any surjective $\mathfrak{C}$-map $\phi\colon R \to S$ and zero-dimensional R-module M, a specific isomorphism

$$\phi_\sharp M \xrightarrow{\ \sim\ } \mathrm{Hom}_R(S, M).$$

The full characterization of $(-)_\sharp$ in 4.3.1 also involves concrete descriptions of some special comparison maps via the specific isomorphisms in (i) and (ii).

(B) Using $(-)_\sharp$, we construct a Coz^0-valued pseudofunctor $(-)^\natural$ on $\mathbb{F}_c$ as follows. For any $\mathbb{F}_c$-map $(\mathcal{X}, \Delta_1) \xrightarrow{f} (\mathcal{Y}, \Delta)$ and points $x \in \mathcal{X}$, $y := f(x) \in \mathcal{Y}$, the natural induced map of complete local rings $\widehat{f_x}\colon \widehat{\mathcal{O}_{\mathcal{Y},y}} \to \widehat{\mathcal{O}_{\mathcal{X},x}}$ (completions

being along the respective maximal ideals) is in $\mathfrak{C}$. For $\mathcal{M}^\bullet \in \mathrm{Coz}^0_\Delta(\mathcal{Y})$ the $\mathcal{O}_{\mathcal{Y},y}$-module $\mathcal{M}^\bullet(y)$, being a zero-dimensional $\mathcal{O}_{\mathcal{Y},y}$-module, is also naturally an $\widehat{\mathcal{O}_{\mathcal{Y},y}}$-module, from which we get the zero-dimensional $\mathcal{O}_{\mathcal{X},x}$-module $\widehat{f}_{x\sharp}(\mathcal{M}^\bullet(y))$. We now let $f^\natural\mathcal{M}^\bullet$ be the unique object in $\mathrm{Coz}^0_{\Delta_1}(\mathcal{X})$ given by

$$(f^\natural\mathcal{M}^\bullet)(x) = \widehat{f}_{x\sharp}(\mathcal{M}^\bullet(y)) \qquad (x \in \mathcal{X},\ y = f(x)). \tag{1}$$

Then $f^\natural$ is a Coz^0-valued functor. Furthermore, pseudofunctoriality of $(-)_\sharp$ induces a pseudofunctor $(-)^\natural$ in an obvious manner.

(C) The upgrading of the Coz^0-valued pseudofunctor $(-)^\natural$ to a Coz-valued one $(-)^\sharp$ is carried out over two subcategories of $\mathbb{F}_\mathrm{c}$, namely, the subcategory of smooth maps and the subcategory of closed immersions. (Dealing with the smooth subcategory is not at all straightforward, see Proposition 5.2.1, but dealing with closed immersions is, see Proposition 6.1.2.)

Let us elaborate. Notation remains as in (B), but now $\mathcal{M}^\bullet \in \mathrm{Coz}_\Delta(\mathcal{Y})$.

Suppose f is smooth. Assume f has constant relative dimension d. (This can be arranged by restricting to connected components if necessary). Then we specify a natural graded isomorphism (see (76))

$$f^\natural\mathcal{M}^\bullet \xrightarrow{\ \sim\ } E_{\Delta_1}\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet \otimes_{\mathcal{Y}} \omega_f[d]). \tag{2}$$

The inputs into the definition of (2) can be organized into two parts. The first involves a specific isomorphism (§5.1), with $r = \dim(\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x})$, $M := \mathcal{M}^\bullet(y)$:

$$\big(E_{\Delta_1}\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet \otimes_{\mathcal{Y}} \omega_f[d])\big)(x) \xrightarrow{\ \sim\ } H^r_{m_x}(M \otimes_y (\omega_f)_x).$$

The second involves an isomorphism

$$H^r_{m_x}(M \otimes_y (\omega_f)_x) \xrightarrow{\ \sim\ } \widehat{f}_{x\sharp}M = (f^\natural\mathcal{M}^\bullet)(x)$$

given up to a sign by completion and the isomorphism of case (i) in step (A) above.

If f is a closed immersion, then we specify a natural graded isomorphism

$$f^\natural\mathcal{M}^\bullet \xrightarrow{\ \sim\ } f^{-1}\mathcal{H}om_{\mathcal{Y}}(f_*\mathcal{O}_{\mathcal{X}}, \mathcal{M}^\bullet). \tag{3}$$

As with (2), the inputs used in defining (3) can be organized into two parts, one consisting of a specific natural isomorphism (with $M = \mathcal{M}^\bullet(y)$)

$$(f^{-1}\mathcal{H}om_{\mathcal{Y}}(f_*\mathcal{O}_{\mathcal{X}}, \mathcal{M}^\bullet))(x) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, M)$$

and the other of an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, M) \xrightarrow{\ \sim\ } \widehat{f}_{x\sharp}M = (f^\natural\mathcal{M}^\bullet)(x)$$

given by completion and the isomorphism of case (ii) in step (A) above.

The graded isomorphisms in (2) and (3) are canonical, so we have a canonical choice for a differential on $f^\natural\mathcal{M}^\bullet$ whenever f is a smooth $\mathbb{F}_\mathrm{c}$-map or a closed immersion, namely the unique one such that (2) or (3) (as the case may be) underlies an isomorphism of complexes. (When f is a smooth closed immersion the two possible differentials coincide, see 8.2.1.) Thus we have a canonical choice for a Coz-valued functor $f^\sharp$ for all f in either of the two subcategories under consideration.

We are now in a position to state our main result. The notation remains as in steps (A)–(C) above. Refer to the beginning of §4 for the notations $C^\#$ (for comparison isomorphism) and $\delta^\#$ (for unit isomorphism) used in the definition of "pseudofunctor." Recall that for any $(\mathcal{X}, \Delta) \in \mathbb{F}_\mathrm{c}$, the forgetful functor

$\mathtt{Fgt}_\Delta(\mathcal{X})\colon \mathrm{Coz}_\Delta(\mathcal{X}) \to \mathrm{Coz}^0_\Delta(\mathcal{X})$ operates by forgetting the differential on a Cousin complex. For convenience, we shall abuse notation and use only the symbol $\mathtt{Fgt}$, the rest being clear from the context.

Main Theorem. *There exists a unique* Coz*-valued pseudofunctor* $(-)^\sharp$ *on* $\mathbb{F}_\mathrm{c}$ *satisfying the following conditions.*

(i) *The forgetful functor* $\mathtt{Fgt}$ *makes* $(-)^\sharp$ *into the above pseudofunctor* $(-)^\natural$. *In other words:*
  (a) *For any* $\mathbb{F}_\mathrm{c}$*-morphism* f, *we have* $\mathtt{Fgt}\circ f^\sharp = f^\natural \circ \mathtt{Fgt}$.
  (b) *For any* $\mathbb{F}_\mathrm{c}$*-morphisms* f, g *such that the composition* gf *is defined it holds that, via* (a), $\mathtt{Fgt}(C^\sharp_{f,g}(-)) = C^\natural_{f,g}(\mathtt{Fgt}(-))$.
  (c) *For any* $(\mathcal{X},\Delta)\in\mathbb{F}_\mathrm{c}$ *it holds that, via* (a), $\mathtt{Fgt}(\delta^\sharp_{\mathcal{X}}(-)) = \delta^\natural_{\mathcal{X}}(\mathtt{Fgt}(-))$.

(ii) *If the* $\mathbb{F}_\mathrm{c}$*-morphism* $f\colon (\mathcal{X},\Delta_1)\to(\mathcal{Y},\Delta)$ *is smooth* (*resp. a closed immersion*) *then for* $\mathcal{M}^\bullet \in \mathrm{Coz}_\Delta(\mathcal{Y})$ *the differential on* $f^\sharp\mathcal{M}^\bullet$ *is the unique one such that the graded isomorphism in* (2) (*resp.* (3)) *underlies an isomorphism of complexes* (*see* (i)(a)).

(iii) *If* $(\mathcal{X},\Delta)\in\mathbb{F}_\mathrm{c}$, $1_\mathcal{X}$ *is the identity map of* $\mathcal{X}$, *and* $u\colon \mathcal{U}\hookrightarrow\mathcal{X}$ *is the inclusion map of an open subset of* $\mathcal{X}$, *then for any* $\mathcal{M}^\bullet\in\mathrm{Coz}_\Delta(\mathcal{X})$, $u^\sharp\mathcal{M}^\bullet$ *is the restriction to* $\mathcal{U}$ *of the complex* $1^\sharp_\mathcal{X}\mathcal{M}^\bullet$.

Remark. The functor from $\mathcal{O}_\mathcal{X}$-modules to $\mathcal{O}_\mathcal{U}$-modules given by "restriction to $\mathcal{U}$," being left-adjoint to the direct image functor u_*, can be identified with u^*. When this is done, the condition in (iii) becomes

$$u^\sharp = u^*\circ 1^\sharp_\mathcal{X}. \tag{iii)$'$}$$

And then if the functor $(-)^\sharp$ is replaced by its normalization (see §4), the condition becomes simply that $u^\sharp = u^*$.

The uniqueness part of the theorem is easy to prove. Indeed if $(-)^\sharp$ and $(-)^{\sharp'}$ are pseudofunctors satisfying conditions (i) and (ii) of the theorem, then for $f,\mathcal{X},\mathcal{Y},\mathcal{M}^\bullet$ as in (ii) of the theorem, by (i)(a) we have $f^\sharp\mathcal{M}^\bullet = f^{\sharp'}\mathcal{M}^\bullet$ as graded torsion modules. It suffices to check locally on $\mathcal{X}$ that the differentials on these graded objects agree, i.e., that for u and $1_\mathcal{X}$ as in (iii), $u^*f^\sharp\mathcal{M}^\bullet = u^*f^{\sharp'}\mathcal{M}^\bullet$ as complexes. A straightforward check of definitions shows that the functorial composition

$$(fu)^\sharp \xrightarrow[C^\sharp_{u,f}]{\sim} u^\sharp f^\sharp \overset{\text{(iii)}'}{=\!=} u^*1^\sharp_\mathcal{X} f^\sharp \xrightarrow[u^*C^\sharp_{1_\mathcal{X},f}]{\sim} u^*f^\sharp$$

is the identity. (In other words, for every $\mathcal{M}^\bullet$ and $x\in\mathcal{U}$, the composition induces the identity map of $[(fu)^\sharp\mathcal{M}^\bullet](x) = \widehat{f}_{x\sharp}(\mathcal{M}^\bullet(f(x))) = [u^*f^\sharp\mathcal{M}^\bullet](x)$.) This can also be stated via "restriction" as

$$(f|_\mathcal{U})^\sharp = f^\sharp|_\mathcal{U}.$$

The same holds for $(-)^{\sharp'}$. Hence we can replace f by fu, and so by 2.4.4 we may assume without loss of generality that f factors as $f = hi$ where h is a smooth $\mathbb{F}_\mathrm{c}$-map and i is a closed immersion. By (ii), $h^\sharp = h^{\sharp'}$ and $i^\sharp = i^{\sharp'}$. By (i)(b) it follows that $f^\sharp\mathcal{M}^\bullet = f^{\sharp'}\mathcal{M}^\bullet$ as complexes too, q.e.d.

Existence, which is the final step of our construction, is the difficult part of the Theorem. For instance, though we have obtained, in step (C), a definition for $f^\sharp$ over the subcategory of smooth $\mathbb{F}_\mathrm{c}$-maps, it is not at all obvious that $(-)^\sharp$ is a pseudofunctor on that subcategory. Even the seemingly simple condition (iii) is not trivial to verify, see Proposition 8.1.1.

(D) First suppose the $\mathbb{F}_c$-map f admits a factorization $f = hi$ where h is a separated smooth $\mathbb{F}_c$-map and i is a closed immersion. Then, using step (C) twice, we get a differential on $h^\natural i^\natural \mathcal{M}^\bullet$. The graded comparison $C^\natural_{i,h}\colon h^\natural i^\natural \mathcal{M}^\bullet \xrightarrow{\sim} f^\natural \mathcal{M}^\bullet$ then induces a differential $d_{h,i}$ on $f^\natural \mathcal{M}^\bullet$. A necessary condition for the existence of $(-)^\sharp$ in the Main Theorem is that this differential on $f^\natural \mathcal{M}^\bullet$ not depend on the choice of factorization $f = hi$. In other words, if there is another factorization $f = h'i'$ with h' smooth and i' a closed immersion then one must have $d_{h,i} = d_{h',i'}$.

In Proposition 8.3.3 we prove that this necessary condition holds. The key inputs are 5.2.1, 6.1.2, 6.2.3 and 7.3.2. Since $\mathbb{F}_c$-maps all have local factorizations as above, the canonical nature of our construction makes it fairly easy to patch the resulting local pseudofunctors together to a global one, see §8.4.

In §9 we discuss residual complexes. We end in §10 with some results on $(-)^\sharp$ for étale maps and on certain Cohen-Macaulay complexes that we encounter while constructing $(-)^\sharp$ over smooth maps.

1.4. Conventions. We use the following notation and conventions in this paper. The Bourbaki dangerous-bend symbol indicates potential sources of ambiguity or conflict with some other convention.

(i) Let $\mathcal{A}$ be an abelian category. Set
  $\mathbf{C}(\mathcal{A})$:=the category of $\mathcal{A}$-complexes,
  $\mathbf{K}(\mathcal{A})$:=the corresponding homotopy category,
  $\mathbf{D}(\mathcal{A})$:=the corresponding derived category.
The differential $d^\bullet_C$ in a complex $C^\bullet$ is always understood to increase degree: d^n_C maps C^n to C^{n+1} for all $n \in \mathbb{Z}$.

(ii) Let $(X, \mathcal{O}_X)$ be a ringed space, and $x \in X$. Set
  $\mathcal{A}(X)$:=the abelian category of $\mathcal{O}_X$-modules,
  $\mathcal{A}_{\mathrm{qc}}(X)$ (resp. $\mathcal{A}_{\mathrm{c}}(X)$, resp. $\mathcal{A}_{\vec{\mathrm{c}}}(X)$) := the full subcategory of $\mathcal{A}(X)$ whose objects are the quasi-coherent (resp. coherent, resp. $\varinjlim$'s of coherent) $\mathcal{O}_X$-modules,
  $\mathcal{E} \otimes_X \mathcal{F}$:= the tensor product of the $\mathcal{O}_X$-modules $\mathcal{E}$, $\mathcal{F}$,
  $E \otimes_x F$:= the tensor product of the $\mathcal{O}_{X,x}$-modules E, F,
  $\mathbf{C}(X) := \mathbf{C}(\mathcal{A}(X)), \quad \mathbf{K}(X) := \mathbf{K}(\mathcal{A}(X)), \quad \mathbf{D}(X) := \mathbf{D}(\mathcal{A}(X))$.
For any full subcategory $\mathcal{A}_?(X)$ of $\mathcal{A}(X)$, let $\mathbf{D}_?(X) \subset \mathbf{D}(X)$ be the full subcategory of $\mathbf{D}_?(X)$ whose objects are the complexes $\mathcal{F}^\bullet$ with homology $H^m(\mathcal{F}^\bullet)$ in $\mathcal{A}_?(X)$, and $\mathbf{D}^+_?(X) \subset \mathbf{D}_?(X)$ (resp. $\mathbf{D}^-_?(X) \subset \mathbf{D}_?(X)$) the full subcategory whose objects are the complexes $\mathcal{F}^\bullet \in \mathbf{D}_?(X)$ such that $H^m(\mathcal{F}^\bullet)$ vanishes for all $m \ll 0$ (resp. $m \gg 0$).

(iii) For a formally smooth local homomorphism $\phi\colon (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of noetherian complete local rings, the usual definition of the relative dimension of ϕ, viz., $\dim(B/\mathfrak{m}_A B)$, is inconsistent with the definition in 2.6.2 of relative dimension for the induced formal scheme-map $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$. The meaning of the term 'relative dimension' will therefore depend on whether the map under consideration is in the category of local rings or that of formal schemes.

(iv) We adopt the usual sign convention for the differential d in the tensor product of two complexes $A^\bullet$, $B^\bullet$ (over rings, ringed spaces, etc.), expressed symbolically by

$$d^n|(A^p \otimes B^{n-p}) = d^p_A \otimes 1 + (-1)^p \otimes d^{n-p}_B.$$

By way of convention for how to connect $\otimes = \otimes_X$ with the translation functor on complexes, noting that $A^{p+i}\otimes B^{q+j}$ occurs as a direct summand in degree $p+q$ of both $A^\bullet[i]\otimes B^\bullet[j]$ and $(A^\bullet\otimes B^\bullet)[i+j]$, we choose the unique isomorphism of complexes

$$\theta=\theta^{A,B}_{i,j}\colon A^\bullet[i]\otimes B^\bullet[j] \xrightarrow{\sim} (A^\bullet\otimes B^\bullet)[i+j] \qquad (i,j\in\mathbb{Z})$$

such that for any $p,q\in\mathbb{Z}$,

$$\theta_{i,j}\big|(A^{p+i}\otimes B^{q+j}) = \text{multiplication by } (-1)^{pj}\ .$$

(v) In particular, (iv) applies when $A^\bullet$ and $B^\bullet$ are concentrated in degree 0. Thus when A and B are $\mathcal{O}_X$-modules, our choice for a natural isomorphism of complexes $A[i]\otimes B[j] \xrightarrow{\sim} (A\otimes B)[i+j]$ is given in degree $-i-j$ by $(-1)^{ij}$ times the identity map of $A\otimes B$. (In (iv), take $p=-i$, $q=-j$.)

(vi) Recall that a δ-functor between two triangulated categories $\mathbf{D}_1,\mathbf{D}_2$, with translation functors T_1,T_2 respectively, is a pair (F,Θ) consisting of an additive functor $F\colon \mathbf{D}_1\to\mathbf{D}_2$ and a natural isomorphism $\Theta\colon FT_1 \xrightarrow{\sim} T_2F$ such that for any triangle $A\xrightarrow{u} B\xrightarrow{v} C\xrightarrow{w} T_1A$ in $\mathbf{D}_1$, the corresponding diagram

$$FA\xrightarrow{Fu} FB\xrightarrow{Fv} FC\xrightarrow{\Theta\,\circ\, Fw} T_2FA$$

is a triangle in $\mathbf{D}_2$. Explicit reference to Θ is frequently suppressed once Θ has been specified.

(vii) With notation as in (iv), for a fixed $B^\bullet\in\mathbf{C}(X)$ where $B^\bullet$ consists of flat $\mathcal{O}_X$-modules, the functor sending $A^\bullet$ to $A^\bullet\otimes B^\bullet$ induces a functor from $\mathbf{D}(X)$ to itself, thus yielding a δ-functor (see (vi)) where Θ is given by the identity map $=\theta_{1,0}$ from (iv). Similarly, if we fix $A^\bullet$ as a complex of flat $\mathcal{O}_X$-modules then the functor sending $B^\bullet$ to $A^\bullet\otimes B^\bullet$, induces a functor from $\mathbf{D}(X)$ to itself which also upgrades to a δ-functor. However in this case $\Theta=\theta_{0,1}$ is *not* the identity map.

(viii) For a complex $\mathcal{F}^\bullet\in\mathbf{C}(\mathcal{A})$ (see (i)), we have $(\mathcal{F}^\bullet[n])^i=\mathcal{F}^{i+n}$, and we can identify the submodule of i-cocycles in $\mathcal{F}^\bullet[n]$ with that of $i+n$-cocycles in $\mathcal{F}^\bullet$, and similarly for coboundaries. Accordingly, we make the identification $H^i(\mathcal{F}^\bullet[n])=H^{i+n}(\mathcal{F}^\bullet)$ without introducing any signs.

(ix) Let $\mathcal{A}$ be an abelian category. For an exact sequence of $\mathcal{A}$-complexes

$$0\longrightarrow A^\bullet\longrightarrow B^\bullet\longrightarrow C^\bullet\longrightarrow 0,$$

the connecting homomorphism $H^iC^\bullet\to H^{i+1}A^\bullet$ is the usual one described via "chasing elements." This connecting map, modulo the identification $H^i(A^\bullet[1])=H^{i+1}A^\bullet$, is (-1) times the map obtained by applying H^i to the derived category map $C^\bullet\to A^\bullet[1]$ in the triangle associated to the above exact sequence.

(x) For a complex $\mathcal{F}^\bullet$ we use the following truncation operators

$$\begin{array}{rcccccccccc}
\sigma_{\ge p}\mathcal{F}^\bullet := & \cdots\longrightarrow & 0 & \longrightarrow & \mathcal{F}^p & \longrightarrow & \mathcal{F}^{p+1} & \longrightarrow & \mathcal{F}^{p+2} & \longrightarrow\cdots\\
\sigma_{\le p}\mathcal{F}^\bullet := & \cdots\longrightarrow & \mathcal{F}^{p-1} & \longrightarrow & \mathcal{F}^p & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\cdots\\
\tau_{\ge p}\mathcal{F}^\bullet := & \cdots\longrightarrow & 0 & \longrightarrow & \operatorname{coker} d^{p-1} & \longrightarrow & \mathcal{F}^{p+1} & \longrightarrow & \mathcal{F}^{p+2} & \longrightarrow\cdots\\
\tau_{\le p}\mathcal{F}^\bullet := & \cdots\longrightarrow & \mathcal{F}^{p-1} & \longrightarrow & \ker d^p & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\cdots
\end{array}$$

where $d^\bullet$ is the differential in $\mathcal{F}^\bullet$; and $\sigma_{>p}:=\sigma_{\ge p+1}$, $\sigma_{<p}:=\sigma_{\le p+1}$, etc.

These operators induce functors $\mathbf{D}(\mathcal{A})\to\mathbf{D}(\mathcal{A})$.

2. Preliminaries on formal schemes

In this section we recall and develop some subsequently-needed basic notions pertaining to formal schemes, their morphisms, and their (sheaves of) modules. For definitions and properties of formal schemes, see [**10**, §10]. *Unless otherwise indicated, all formal schemes will be understood to be noetherian.*

In §2.1 we define *essentially pseudo-finite-type maps* of formal schemes, a mild generalization of the notion of pseudo-finite-type maps treated in [**2**] and [**27**]; and we discuss the behavior of *codimension functions.* In §2.2 we discuss *quasi-coherent modules* over formal schemes, especially in relation to modules over noetherian adic rings. In §2.3 we discuss *torsion modules* over formal schemes, an important class which includes arbitrary modules over an ordinary scheme. In §2.4 we discuss *smooth maps*—those which are formally smooth and essentially of pseudo-finite type. A key property (2.4.4) of essentially pseudo-finite-type maps is that they factor locally as (smooth) $\circ$ (closed immersion). In the remaining two subsections we discuss *modules of continuous differentials* relative to essentially pseudo-finite-type maps, first of noetherian adic rings, and then of formal schemes.

2.1. Codimension functions and maps of formal schemes. A topological ring R is *adic* if there exists an R-ideal I whose powers form a base of neighborhoods of (0), and if R is complete and Hausdorff. Any such I is called an *ideal of definition of* R (or a *defining ideal of* R). If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a formal scheme then there exists a coherent $\mathcal{O}_{\mathcal{X}}$-ideal $\mathcal{I}$ whose sections over any affine open set $\mathcal{U}$ of $\mathcal{X}$ form a defining ideal for the adic ring $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$; any such $\mathcal{I}$ is called an *ideal of definition of* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ (or simply of $\mathcal{X}$) ([**10**, §§10.5, 10.10]). The ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is then a *noetherian ordinary scheme,* that is, a noetherian formal scheme with discretely topologized structure sheaf, or equivalently, with (0) as a defining ideal.

The notion in [**10**, §10.13] of finite-type maps of formal schemes is not adequate for our purposes, and instead we consider a slight generalization of the notion in [**2**] and [**27**] of pseudo-finite type maps. A homomorphism $f\colon A \to B$ of noetherian adic rings is (*essentially*) *of pseudo-finite type* if it is continuous and if for one—hence any—defining ideal $\mathfrak{b} \subset B$ the composition $A \to B \to B/\mathfrak{b}$ is (essentially) of finite type, i.e., $B/\mathfrak{b}$ is (a localization of) a finite-type A-algebra.

A map of not-necessarily-noetherian ordinary schemes $f_0\colon X \to Y$ is *essentially of finite type* if every $y \in Y$ has an affine open neighborhood $V = \mathrm{Spec}(A)$ such that $f_0^{-1}V$ can be covered by finitely many affine open $U_i = \mathrm{Spec}(C_i)$ such that the corresponding maps $A \to C_i$ are essentially of finite type. For any morphism of formal schemes $f\colon (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, there exist ideals of definition $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ and $\mathcal{J} \subset \mathcal{O}_{\mathcal{Y}}$ satisfying $\mathcal{J}\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$ ([**10**, 10.6.10]); and correspondingly there is an induced map of ordinary schemes $f_0\colon (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\mathcal{J})$ ([**10**, 10.5.6]). We say f is (*essentially*) *of pseudo-finite type* if f_0 is (essentially) of finite type.

This property of maps is independent of the choice of defining ideals $\mathcal{I}, \mathcal{J}$. It behaves well with respect to composition and base change: if $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are formal-scheme maps, and if both f and g are essentially of pseudo-finite type, then so is the composition gf; and if gf and g are essentially of pseudo-finite type then so is f; moreover, if $f\colon \mathcal{X} \to \mathcal{Y}$ is essentially of pseudo-finite type and $\mathcal{Y}' \to \mathcal{Y}$ is any map of formal schemes, then $\mathcal{X}' := \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ is noetherian and the projection $\mathcal{X}' \to \mathcal{Y}'$ is essentially of pseudo-finite type (cf. proof of 2.1.3).

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We don't know for an arbitrary formal-scheme map $\mathcal{U} = \mathrm{Spf}(B) \to \mathcal{V} = \mathrm{Spf}(A)$ that is essentially of pseudo-finite type, whether the corresponding g $A \to B$ is essentially of pseudo-finite type.

A formal scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is called (*universally*) *catenary* if there exists one defining ideal $\mathcal{I}$ such that the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is (universally) catenary. This implies that for *any* defining ideal $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$ the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J})$ is (universally) catenary—an easy consequence of the fact that on a formal scheme, for any two defining ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ there exists an integer n such that $\mathcal{I}_1^n \subset \mathcal{I}_2$.

A *codimension function* on the underlying topological space $\mathcal{X}$ of a formal scheme is a function $\Delta\colon \mathcal{X} \to \mathbb{Z}$ such that $\Delta(x') = \Delta(x) + 1$ for every immediate specialization $x \rightsquigarrow x'$ of points in $\mathcal{X}$.[1] If $\mathcal{X}$ admits a codimension function, then $\mathcal{X}$ has to be catenary. When $\mathcal{X}$ is connected and noetherian, any two codimension functions on $\mathcal{X}$ differ by a constant.

EXAMPLE 2.1.1. If $\mathcal{X}$ is catenary and irreducible, then for any integer n there is a unique codimension function on $\mathcal{X}$ assigning n to the generic point. If $\mathcal{X}$ is catenary and biequidimensional ([**8**, 14.3.3]) then we can assign one fixed integer to all the closed points of $\mathcal{X}$, and this extends uniquely to a codimension function.

EXAMPLE 2.1.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of formal schemes that is essentially of pseudo-finite type. If $\mathcal{Y}$ is universally catenary then so is $\mathcal{X}$. Furthermore, if $\mathcal{Y}$ admits a codimension function Δ then *the function $f^\sharp\Delta$ on $\mathcal{X}$ such that*

$$(f^\sharp\Delta)(x) = \Delta(y) - \mathrm{tr.deg.}_{k(y)}k(x) \qquad (x \in \mathcal{X},\ y := f(x))$$

(where $\mathrm{tr.deg.}_{k(y)}k(x)$ is the transcendence degree of the residue field extension $k(x)/k(y)$) *is a codimension function on* $\mathcal{X}$, as follows from the dimension formula ([**8**, 5.6.2]).

In particular, from 2.1.2 we see that any formal scheme that is essentially of pseudo-finite type over a field k admits a codimension function. Furthermore if $f\colon \mathcal{X} \to \mathrm{Spec}(k)$ is essentially of pseudo-finite type and if Δ is the function on $\mathrm{Spec}(k)$ sending the unique point to 0 then $f^\sharp\Delta(x) = 0$ if and only if x is a closed point of $\mathcal{X}$.

Let $\mathbb{F}$ be the category whose objects are the (noetherian) universally catenary formal schemes admitting a codimension function, and whose morphisms are the formal-scheme maps which are essentially of pseudo-finite type. Let $\mathbb{F}_{\mathrm{c}}$ be the category whose objects are the pairs $(\mathcal{X}, \Delta)$ with $\mathcal{X} \in \mathbb{F}$ and Δ a codimension function on $\mathcal{X}$, and whose morphisms $(\mathcal{X}', \Delta') \to (\mathcal{X}, \Delta)$ are the $\mathbb{F}$-morphisms $f\colon \mathcal{X}' \to \mathcal{X}$ such that $\Delta' = f^\sharp\Delta$. (Note that the formula in 2.1.2 behaves well with respect to compositions in $\mathbb{F}$.) The next Lemma is about fiber products in these categories. (See [**10**, 10.7.3] for the construction of fiber products of formal schemes).

LEMMA 2.1.3. (a) *If $f\colon \mathcal{X} \to \mathcal{Z}$, $g\colon \mathcal{Y} \to \mathcal{Z}$ are $\mathbb{F}$-morphisms and $\mathcal{W} := \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, with projections $\mathcal{W} \xrightarrow{p} \mathcal{X}$, $\mathcal{W} \xrightarrow{q} \mathcal{Y}$, then $(\mathcal{W}, p, q)$ is an $\mathbb{F}$-fiber product of f and g.*

(b) *Moreover, if Δ is a codimension function on $\mathcal{Z}$, then $p^\sharp f^\sharp\Delta = q^\sharp g^\sharp\Delta =$* (say) $\Delta^\times$, *and $((\mathcal{W}, \Delta^\times), p, q)$ is an $\mathbb{F}_{\mathrm{c}}$-fiber product of $f\colon (\mathcal{X}, f^\sharp\Delta) \to (\mathcal{Z}, \Delta)$ and $g\colon (\mathcal{Y}, g^\sharp\Delta) \to (\mathcal{Z}, \Delta)$.*

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[1] A specialization $x \rightsquigarrow x'$—i.e., x' is in the closure of $\{x\}$—is called *immediate* if $x \neq x'$ and there are no specializations $x \rightsquigarrow x'' \rightsquigarrow x'$ with $x \neq x'' \neq x'$.

PROOF. Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be defining ideals for $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively, so that $\mathcal{W}$ has the defining ideal $\mathcal{L} := \mathcal{I}\mathcal{O}_{\mathcal{W}} + \mathcal{J}\mathcal{O}_{\mathcal{W}}$. If $\mathcal{X}_0, \mathcal{Y}_0, \mathcal{Z}_0, \mathcal{W}_0$ are the schemes obtained from $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}$ by going modulo the corresponding defining ideals $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$, then $\mathcal{W}_0 := \mathcal{X}_0 \times_{\mathcal{Z}_0} \mathcal{Y}_0$. The natural map $\mathcal{W}_0 \to \mathcal{Y}_0$ is essentially of finite type, so the induced map $q\colon \mathcal{W} \to \mathcal{Y}$ is essentially of pseudo-finite type. Moreover $\mathcal{W}_0$ is noetherian, hence so is $\mathcal{W}$ ([**10**, 10.6.4]). Now 2.1.2 shows that $\mathcal{W} \in \mathbb{F}$, and the rest is straightforward. $\square$

2.2. Quasi-coherent modules. We recall some basic facts about modules over formal schemes (assumed, as always, to be noetherian). Proofs can be found in [**27**, §3], or [**2**, §3].

Let A be a noetherian adic ring and $\mathcal{U} := \operatorname{Spf}(A)$ the corresponding affine formal scheme. Let $\operatorname{Mod}(A)$ (resp. $\operatorname{Mod}_f(A)$) denote the category of A-modules (resp. finitely generated A-modules). For any A-module M, consider the presheaf that assigns to any open set $\mathcal{V}$, the module $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{U}}) \otimes_A M$. Let $M^{\sim_A}$ (or $M^{\sim}$ when there is no cause for confusion) denote the associated sheaf. This defines a functor $\sim_A\colon \operatorname{Mod}(A) \to \mathcal{A}(\mathcal{U})$ (see (ii) of §1.4).

In fact, if $\kappa = \kappa_A\colon \mathcal{U} \to U := \operatorname{Spec}(A)$ is the canonical map, and $\widetilde{M}$ is the sheaf over $\operatorname{Spec}(A)$ corresponding to M, then there is a functorial isomorphism

$$M^{\sim} \xrightarrow{\ \sim\ } \kappa^* \widetilde{M}.$$

In other words, $M^{\sim}$ represents the functor $\operatorname{Hom}_{\mathcal{O}_U}(\widetilde{M}, \kappa_* \mathcal{G})$ of $\mathcal{O}_{\mathcal{U}}$-modules $\mathcal{G}$. Indeed, an $\mathcal{O}_{\mathcal{U}}$-homomorphism of sheaves $M^{\sim} \to \mathcal{G}$ corresponds naturally to an $\mathcal{O}_{\mathcal{U}}$-homomorphism of presheaves $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{U}}) \otimes_A M \to \Gamma(\mathcal{V}, \mathcal{G})$, which is determined by the single A-homomorphism $M \to \Gamma(\mathcal{U}, \mathcal{G})$ obtained by taking $\mathcal{V} = \mathcal{U}$, and thus, in view of [**10**, p. 213, Cor. (1.7.4)], we have natural isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{U}}}(M^{\sim}, \mathcal{G}) \cong \operatorname{Hom}_A(M, \Gamma(\mathcal{U}, \mathcal{G})) \cong \operatorname{Hom}_{\mathcal{O}_U}(\widetilde{M}, \kappa_* \mathcal{G}).$$

The first of these isomorphisms shows, moreover, that *the functor* $\sim_A$ *is left-adjoint to the functor* $\Gamma(\mathcal{U}, -)$ of $\mathcal{O}_{\mathcal{U}}$-modules.[2]

PROPOSITION 2.2.1 ([**27**, p. 874, Prop. 3.2], [**2**, p. 31, Prop. 3.1.1]). *With the preceding notation, and* $\sim := \sim_A$,

(i) *The functor* $\sim$ *is exact and commutes with direct limits.*
(ii) *For any A-module M, $M^{\sim} \in \mathcal{A}_{\vec{c}}(\mathcal{U})$ and is quasi-coherent.*
(iii) *The functor* $\sim$ *is an equivalence of categories between* $\operatorname{Mod}(A)$ *and* $\mathcal{A}_{\vec{c}}(\mathcal{U})$, *with quasi-inverse* $\Gamma(\mathcal{U}, -)\colon \mathcal{A}_{\vec{c}}(\mathcal{U}) \to \operatorname{Mod}(A)$. *These quasi-inverse equivalences restrict to quasi-inverse equivalences between* $\operatorname{Mod}_f(A)$ *and* $\mathcal{A}_{\mathrm{c}}(\mathcal{U})$.
(iv) *For any affine open $\mathcal{V} \subset \mathcal{U}$ and any A-module M, the natural map is an isomorphism*

$$\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{U}}) \otimes_A M \xrightarrow{\ \sim\ } \Gamma(\mathcal{V}, M^{\sim}).$$

[2] These considerations hold for *any* map of ringed spaces $\varkappa\colon X \to \operatorname{Spec}(A)$. In that generality, 2.2.1(ii) below (with $\vec{c}$ meaning "$\varinjlim$ of finitely-presented") and 2.2.4 (with f replaced by $\varkappa$) hold; and if $\varkappa$ is *flat* then so do 2.2.1(i), remark (a) following 2.2.1, and 2.2.3. What distinguishes κ from the horde is 2.2.1(iii).

Remarks. (a) We can (and will) identify $A^{\sim}$ with $\mathcal{O}_{\mathfrak{U}}$; and for any A-ideal I we can (and will) identify $I^{\sim}$ with its natural image in $A^{\sim}$, so that $I^{\sim}$ will be regarded as an $\mathcal{O}_{\mathfrak{U}}$-ideal. In view of (i), we will also identify $(A/I)^{\sim}$ with $\mathcal{O}_{\mathfrak{U}}/I^{\sim}$.

(b) If I is a defining ideal of the adic ring A then $I^{\sim}$ is a defining ideal of $\mathfrak{U}$.

(c) For any formal scheme $\mathfrak{X}$, (ii) and (iii) in 2.2.1 imply that $\mathcal{A}_{\vec{c}}(\mathfrak{X}) \subset \mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$, as it is enough to check this locally. Equality holds if $\mathfrak{X}$ is an ordinary scheme.

(d) When M is finitely generated, $M^{\sim}$ is the same as M^{Δ} in [**10**, §10.10], so the second part of (iii) is identical with *loc. cit.,* Thm. (10.10.2).

(e) Note that (iv) follows from (iii), because with $B := \Gamma(\mathfrak{V}, \mathcal{O}_{\mathfrak{U}})$ we have $(B \otimes_A M)^{\sim_B} = M^{\sim_A}|_{\mathfrak{V}}$.

Proposition 2.2.2 ([**27**, p. 875, Cor. 3.4], or [**2**, p. 32, Cor. 3.1.4]). *Let $\mathfrak{X}$ be a formal scheme and $x \in \mathfrak{X}$. For any quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ there exists an affine open neighborhood $\mathfrak{U} := \mathrm{Spf}(A)$ of x such that the natural map is an isomorphism $\Gamma(\mathfrak{U}, \mathcal{M})^{\sim_A} \xrightarrow{\sim} \mathcal{M}\big|_{\mathfrak{U}}$.*

Lemma 2.2.3. *Let $\mathfrak{U} := \mathrm{Spf}(A)$ be an affine formal scheme and let M, N be A-modules. Set $\sim := \sim_A$.*

(i) *If N is finitely generated then the map φ corresponding to the natural map $\mathrm{Hom}_A(N, M) \to \mathrm{Hom}_{\mathcal{O}_{\mathfrak{U}}}(N^{\sim}, M^{\sim})$ is an isomorphism*

$$\mathrm{Hom}_A(N, M)^{\sim} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}(N^{\sim}, M^{\sim}).$$

(ii) *The map corresponding to the natural map*

$$M \otimes_A N \cong M^{\sim}(\mathfrak{U}) \otimes_A N^{\sim}(\mathfrak{U}) \longrightarrow (M^{\sim} \otimes_{\mathcal{O}_{\mathfrak{U}}} N^{\sim})(\mathfrak{U})$$

is an isomorphism

$$(M \otimes_A N)^{\sim} \xrightarrow{\sim} M^{\sim} \otimes_{\mathcal{O}_{\mathfrak{U}}} N^{\sim}.$$

(iii) *For any A-ideals I, J the $\mathcal{O}_{\mathfrak{U}}$-ideals $(IJ)^{\sim}$ and $I^{\sim}J^{\sim}$ coincide.*

(iv) *For any $m > 0$ the natural map is an isomorphism*

$$\Big(\bigwedge_A^m M\Big)^{\sim} \xrightarrow{\sim} \bigwedge_{\mathcal{O}_{\mathfrak{U}}}^m M^{\sim}.$$

Proof. (i). Let $A^i \to A^j \to N \to 0$ be a presentation of N. In the following natural commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_A(N, M)^{\sim} & \longrightarrow & \mathrm{Hom}_A(A^j, M)^{\sim} & \longrightarrow & \mathrm{Hom}_A(A^i, M)^{\sim} \\
& {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} \\
0 \longrightarrow & \mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}(N^{\sim}, M^{\sim}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}(A^{j\sim}, M^{\sim}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}(A^{i\sim}, M^{\sim})
\end{array}$$

the rows are exact by 2.2.1(i). Since $\sim$ commutes with direct sums, the functors Hom_A and $\mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}$ commute with finite direct sums, and since the natural map $\mathrm{Hom}_A(A, M)^{\sim} \to \mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}(A^{\sim}, M^{\sim})$ is an isomorphism, the two vertical arrows on the right are isomorphisms, whence so is φ.

(ii). Proceed as in (i), using a presentation of N; or simply use that with $Q = M$ or N, one has $Q^{\sim} = \kappa^*(\widetilde{Q})$.

(iii). Since $\sim$ is exact, and so "commutes with image," the assertion amounts to equality of the images of the natural maps μ and $\hat{\mu}$ in the following diagram:

$$\begin{tikzcd} (I \otimes_A J)^{\sim} \arrow[r, "\mu"] \arrow[d, "\cong"', "\text{(ii)}"] & A^{\sim} \arrow[d, equal] \\ I^{\sim} \otimes_{\mathcal{O}_{\mathfrak{U}}} J^{\sim} \arrow[r, "\hat{\mu}"'] & \mathcal{O}_{\mathfrak{U}} \end{tikzcd}$$

It is clear that the diagram commutes, whence the conclusion.

(iv) The assertion results, upon passage to associated sheaves, from the natural presheaf isomorphisms (for affine open $\mathcal{V} \subset \mathfrak{U}$):

$$\Gamma(\mathcal{V}, \mathcal{O}_{\mathfrak{U}}) \otimes_A \bigwedge^m M \cong \bigwedge^m \big(\Gamma(\mathcal{V}, \mathcal{O}_{\mathfrak{U}}) \otimes_A M\big) \underset{2.2.1(\text{iv})}{\xrightarrow{\ \sim\ }} \bigwedge^m \big(\Gamma(\mathcal{V}, M^{\sim})\big).$$

$\square$

Lemma 2.2.4. *Let* $\mathcal{V} := \mathrm{Spf}(B) \xrightarrow{f} \mathfrak{U} := \mathrm{Spf}(A)$ *be a map of formal schemes. Then for any* A*-module* M*, the map* ϕ *corresponding to the natural composition*

$$\begin{aligned} M \otimes_A B \xrightarrow{\ \sim\ } \Gamma(\mathfrak{U}, M^{\sim_A}) \otimes_A B &\to \Gamma(\mathfrak{U}, f_* f^*(M^{\sim_A})) \otimes_A B \\ &\to \Gamma(\mathcal{V}, f^*(M^{\sim_A})) \otimes_A B \to \Gamma(\mathcal{V}, f^*(M^{\sim_A})) \end{aligned}$$

is an isomorphism

$$(M \otimes_A B)^{\sim_B} \xrightarrow{\ \sim\ } f^*(M^{\sim_A}).$$

Proof. Let $A^I \to A^J \to M \to 0$ be a presentation of M. In the following diagram of natural induced maps, the top row is obtained by applying $(-\otimes_A B)^{\sim_B}$ to the presentation of M while the bottom row is obtained by applying $f^*(-^{\sim_A})$.

$$\begin{tikzcd} (B^I)^{\sim_B} \arrow[r] \arrow[d, "\cong"'] & (B^J)^{\sim_B} \arrow[r] \arrow[d, "\cong"'] & (M \otimes_A B)^{\sim_B} \arrow[r] \arrow[d, "\phi"] & 0 \\ f^*(A^I)^{\sim_A} \arrow[r] & f^*(A^J)^{\sim_A} \arrow[r] & f^* M^{\sim_A} \arrow[r] & 0 \end{tikzcd}$$

The rows are exact, by 2.2.1(i). Since $\sim_A$, $\sim_B$ and f^* commute with direct sums, and the natural map $(A \otimes_A B)^{\sim_B} \longrightarrow f^*(A^{\sim_A})$ is an isomorphism, the vertical maps on the left are isomorphisms and hence so is ϕ.

Alternatively, since f^* is left-adjoint to f_* one can extract the assertion from the sequence of natural isomorphisms (with $\mathcal{G} \in \mathcal{A}(\mathcal{V})$)

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\mathcal{V}}}\big((M \otimes_A B)^{\sim_B}, \mathcal{G}\big) &\xrightarrow{\ \sim\ } \mathrm{Hom}_B\big(M \otimes_A B, \Gamma(\mathcal{V}, \mathcal{G})\big) \\ &\xrightarrow{\ \sim\ } \mathrm{Hom}_A\big(M, \Gamma(\mathfrak{U}, f_*\mathcal{G})\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_{\mathfrak{U}}}(M^{\sim_A}, f_*\mathcal{G}). \end{aligned}$$

$\square$

Following [**17**, (1.9.1)], we say that a subcategory $\mathcal{A}_?(\mathfrak{X}) \subset \mathcal{A}(\mathfrak{X})$ is *plump* if it is full and if for every exact sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M} \to \mathcal{M}_3 \to \mathcal{M}_4$ in $\mathcal{A}(\mathfrak{X})$ with $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and $\mathcal{M}_4$ in $\mathcal{A}_?(\mathfrak{X})$, $\mathcal{M}$ is in $\mathcal{A}_?(\mathfrak{X})$ too. Then $\mathcal{A}_?(\mathfrak{X})$ is an *abelian* subcategory of $\mathcal{A}(\mathfrak{X})$. Moreover, $\mathbf{D}_?(\mathfrak{X})$ is a *triangulated* subcategory of $\mathbf{D}(\mathfrak{X})$.

(However, the natural functor $\mathbf{D}(\mathcal{A}_?(\mathfrak{X})) \to \mathbf{D}_?(\mathfrak{X})$ need not be an equivalence.)

Proposition 2.2.5 ([**2**, p. 34, Prop. 3.2.2]). *For any formal scheme $\mathfrak{X}$, the subcategories $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ and $\mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ of $\mathcal{A}(\mathfrak{X})$ are plump.*

Let $\mathfrak{U} = \mathrm{Spf}(A)$ be an affine formal scheme and x a point in $\mathfrak{U}$. Let $\mathfrak{p}$ be the open prime ideal in the adic ring A corresponding to x and let m_x denote the maximal ideal of the local ring $\mathcal{O}_{\mathfrak{U},x}$. Let $\mathfrak{a}$ be a defining ideal in A and $\mathcal{I} := \mathfrak{a}^{\sim_A}$ the corresponding defining ideal in $\mathcal{O}_{\mathfrak{U}}$. Recall that $\mathcal{O}_{\mathfrak{U},x}$ is a direct limit of the rings $A_{\{f\}}$ for $f \notin \mathfrak{p}$ where $A_{\{f\}}$ is the completion of A_f along $\mathfrak{a}_f$. In particular there is a natural map $A_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{U},x}$. For any open ideal J in A, with $\mathcal{J} := J^{\sim_A}$, the natural induced map $A_{\mathfrak{p}}/J_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{U},x}/\mathcal{J}_x$ is an isomorphism. Since any power of $J, \mathcal{J}$ is open, there results a canonical isomorphism $(A_{\mathfrak{p}}, J_{\mathfrak{p}})\widehat{\ } \xrightarrow{\sim} (\mathcal{O}_{\mathfrak{U},x}, \mathcal{J}_x)\widehat{\ }$. In particular, we can associate canonically to $x \in \mathfrak{U}$ the following faithfully flat inclusions of noetherian local rings

$$A_{\mathfrak{p}} \hookrightarrow \mathcal{O}_{\mathfrak{U},x} \hookrightarrow B \hookrightarrow C,$$

where $B = (A_{\mathfrak{p}}, \mathfrak{a}A_{\mathfrak{p}})\widehat{\ } \cong (\mathcal{O}_{\mathfrak{U},x}, \mathcal{I}_x)\widehat{\ }$ and $C = (A_{\mathfrak{p}}, \mathfrak{p}A_{\mathfrak{p}})\widehat{\ } \cong (\mathcal{O}_{\mathfrak{U},x}, m_x)\widehat{\ } \cong (B, m_B)\widehat{\ }$.

2.3. Torsion modules. Let A be a ring, $\mathfrak{a}$ an A-ideal. For any A-module M, let $\varGamma_{\mathfrak{a}} M$ be the submodule consisting of those elements which are annihilated by a power of $\mathfrak{a}$. Note that $\varGamma_{\mathfrak{a}} M$ is naturally isomorphic to $\varinjlim_n \mathrm{Hom}_A(A/\mathfrak{a}^n, M)$. We say that M is an $\mathfrak{a}$-*torsion module* if $\varGamma_{\mathfrak{a}} M = M$. If A is a topological ring whose topology is defined by powers of $\mathfrak{a}$ then an $\mathfrak{a}$-torsion module is referred to simply as a *torsion module.*

Lemma 2.3.1. *Let A be a noetherian adic ring and M a torsion A-module. For any multiplicatively closed set $S \subset A$ let $A\{S^{-1}\}$ be the completion of the localization A_S along a defining A-ideal. Then the natural map is an isomorphism*

$$M_S \xrightarrow{\sim} M \otimes_A A\{S^{-1}\}.$$

Proof. Let N be a finitely generated submodule of M. Then the torsion module N is annihilated by some defining ideal, say $\mathfrak{a}$, in A. Hence we have the following isomorphisms

$$N_S \xrightarrow{\sim} N \otimes_A A_S/(\mathfrak{a}) \xrightarrow{\sim} N \otimes_A A\{S^{-1}\}/(\mathfrak{a}) \xleftarrow{\sim} N \otimes_A A\{S^{-1}\}.$$

Taking direct limit over all finitely-generated submodules of M gives the result. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-ideal $\mathcal{I}$ and any $\mathcal{M} \in \mathcal{A}(X)$, set

$$\varGamma_{\mathcal{I}} \mathcal{M} := \varinjlim_{n} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}).$$

We regard $\varGamma_{\mathcal{I}}$ as a subfunctor of the identity functor on $\mathcal{O}_X$-modules. As such it is idempotent and left exact. Since taking direct limits commutes with restriction to open $\mathfrak{U} \subset X$, therefore $(\varGamma_{\mathcal{I}} \mathcal{M})|_{\mathfrak{U}} \cong \varGamma_{\mathcal{I}|_{\mathfrak{U}}}(\mathcal{M}|_{\mathfrak{U}})$.

For a formal scheme $\mathfrak{X}$ with ideal of definition $\mathcal{I}$, we set $\varGamma'_{\mathfrak{X}} := \varGamma_{\mathcal{I}}$, this definition being independent of the choice of the defining ideal $\mathcal{I}$. We call $\mathcal{M} \in \mathcal{A}(X)$ a *torsion $\mathcal{O}_{\mathfrak{X}}$-module* if $\varGamma'_{\mathfrak{X}}\mathcal{M} = \mathcal{M}$. Thus for any $\mathcal{N} \in \mathcal{A}(\mathfrak{X})$, $\varGamma'_{\mathfrak{X}}\mathcal{N}$ is the largest torsion submodule of $\mathcal{N}$. Note that $\mathcal{N}$ is a torsion $\mathcal{O}_{\mathfrak{X}}$-module if and only if there is an open cover $\{\mathfrak{U}_i\}$ of $\mathfrak{X}$ for which each $\mathcal{N}|_{\mathfrak{U}_i}$ is a torsion $\mathcal{O}_{\mathfrak{U}_i}$-module.

Lemma 2.3.2. *Let $\mathcal{X}$ be a formal scheme. For any injective $\mathcal{A}(\mathcal{X})$-module $\mathcal{L}$, $\varGamma'_{\mathcal{X}}\mathcal{L}$ is a flasque sheaf.*

Proof. For any defining ideal $\mathcal{I}$ of $\mathcal{O}_{\mathcal{X}}$, $\varGamma'_{\mathcal{X}}\mathcal{L}$ is a direct limit over the noetherian space $\mathcal{X}$ of the flasque sheaves $\mathcal{H}om(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n, \mathcal{L})$, and so is flasque. □

Lemma 2.3.3. *Let $\mathcal{U} = \mathrm{Spf}(A)$ be an affine formal scheme, and let M be an A-module. Let I be an A-ideal and $\mathcal{I} := I^\sim$ the corresponding $\mathcal{O}_{\mathcal{U}}$-ideal. Then there is a natural isomorphism*

$$\varGamma_{\mathcal{I}}\, M^\sim \xrightarrow{\ \sim\ } (\varGamma_I\, M)^\sim,$$

whose composition with the natural injection $j\colon (\varGamma_I\, M)^\sim \to M^\sim$ is the inclusion $i\colon \varGamma_{\mathcal{I}}\, M^\sim \hookrightarrow M^\sim$. In particular, M is a torsion A-module $\iff$ $M^\sim$ is a torsion $\mathcal{O}_{\mathcal{U}}$-module.

Proof. By remark (a) following 2.2.1, and by 2.2.3(iii), one can identify $(A/I^n)^\sim$ and $\mathcal{O}_{\mathcal{U}}/\mathcal{I}^n$. One checks then that the composed isomorphism

$$\begin{aligned}\varGamma_{\mathcal{I}}\, M^\sim = \varinjlim_n \mathcal{H}om_{\mathcal{O}_{\mathcal{U}}}(\mathcal{O}_{\mathcal{U}}/\mathcal{I}^n, M^\sim) &\underset{2.2.3}{\xrightarrow{\ \sim\ }} \varinjlim_n \mathrm{Hom}_A(A/I^n, M)^\sim \\ &\underset{2.2.1}{\xrightarrow{\ \sim\ }} (\varinjlim_n \mathrm{Hom}_A(A/I^n, M))^\sim = (\varGamma_I\, M)^\sim\end{aligned}$$

has the required properties. In particular, i is an isomorphism iff j is.

Taking I to be an ideal of definition in A, so that $\mathcal{I}$ is an ideal of definition in $\mathcal{O}_{\mathcal{U}}$ (remark (b) following 2.2.1), one has that i is an isomorphism iff $M^\sim$ is a torsion $\mathcal{O}_{\mathcal{U}}$-module, and by 2.2.1(iii), j is an isomorphism iff M is a torsion A-module. The last assertion results. □

For a formal scheme $\mathcal{X}$, $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$ denotes the full subcategory of $\mathcal{A}(\mathcal{X})$ whose objects are the torsion modules; and $\mathcal{A}_{\mathrm{qct}}(\mathcal{X}) := \mathcal{A}_{\mathrm{qc}}(\mathcal{X}) \cap \mathcal{A}_{\mathrm{t}}(\mathcal{X})$ (respectively $\mathcal{A}_{\mathrm{ct}}(\mathcal{X}) := \mathcal{A}_{\mathrm{c}}(\mathcal{X}) \cap \mathcal{A}_{\mathrm{t}}(\mathcal{X})$) denotes the full subcategory of $\mathcal{A}(\mathcal{X})$ whose objects are the quasi-coherent torsion modules (resp. coherent torsion modules).

If $\mathcal{X}$ is an ordinary scheme (i.e., (0) is a defining ideal), then $\mathcal{A}_{\mathrm{t}}(\mathcal{X}) = \mathcal{A}(\mathcal{X})$ and $\mathcal{A}_{\mathrm{qct}}(\mathcal{X}) = \mathcal{A}_{\mathrm{qc}}(\mathcal{X}) = \mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})$.

Proposition 2.3.4. *With the preceding notation*:

(i) *For any $\mathcal{M} \in \mathcal{A}_{\mathrm{qc}}(\mathcal{X})$, it holds that $\varGamma'_{\mathcal{X}}\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$.*

(ii) *The subcategories $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$ and $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ of $\mathcal{A}(\mathcal{X})$ are plump, and hence are abelian categories.*

(iii) *$\mathcal{A}_{\mathrm{qct}}(\mathcal{X}) \subset \mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})$. So if $\mathcal{X}$ is affine, say $\mathcal{X} = \mathrm{Spf}(A)$, and $\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ then by* 2.2.1(iii), *the natural map is an isomorphism $\Gamma(\mathcal{U}, \mathcal{M})^{\sim_A} \xrightarrow{\ \sim\ } \mathcal{M}$.*

Proof. See [**2**, §5.1]. □

We consider next a class of torsion modules that includes the injectives in the categories $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ and $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$. These torsion modules will be the building blocks of the objects of main concern to us, viz., Cousin complexes.

For an abelian group G and a point x of the formal scheme $\mathcal{X}$, let $i_{\mathcal{X},x}G$ (or simply i_xG if no confusion results) be the sheaf whose sections over any open $\mathcal{U} \subset \mathcal{X}$ are the elements of G if $x \in \mathcal{U}$ and (0) otherwise (restriction maps being the obvious ones). For an $\mathcal{O}_{\mathcal{X},x}$-module M the sheaf i_xM has a natural $\mathcal{O}_{\mathcal{X}}$-module structure. Our interest lies in the situation where M is a zero-dimensional $\mathcal{O}_{\mathcal{X},x}$-module.

Lemma 2.3.5. *Let $\mathcal{X}$ be a formal scheme, $x \in \mathcal{X}$, m_x the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$ and M an $\mathcal{O}_{\mathcal{X},x}$-module. The following are equivalent.*

(i) *M is a zero-dimensional $\mathcal{O}_{\mathcal{X},x}$-module, i.e., M is m_x-torsion.*
(ii) *$i_xM \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$.*
(iii) *For every affine open neighborhood $\mathcal{U} = \mathrm{Spf}(A)$ of x, $i_xM|_{\mathcal{U}} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{U})$. Moreover, M being an A-module via the natural map $A \to \mathcal{O}_{\mathcal{X},x}$, there is a natural isomorphism $i_xM|_{\mathcal{U}} \xrightarrow{\sim} M^{\sim}$.*
(iv) *There is an affine open neighborhood $\mathcal{U}$ of x such that $i_xM|_{\mathcal{U}} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{U})$.*

Proof. We show that (i) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (ii), and (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

(i) $\Rightarrow$ (iii). Let $\mathcal{U} = \mathrm{Spf}(A)$ be an affine open neighborhood of x, and for any $f \in A$, $\mathcal{U}_f \subset \mathcal{U}$ the open subset $\mathrm{Spf}(A_{\{f\}})$. We claim that *if $x \notin \mathcal{U}_f$ then $\Gamma(\mathcal{U}_f, M^{\sim}) = 0$, and if $x \in \mathcal{U}_f$ then the natural map $M \to \Gamma(\mathcal{U}_f, M^{\sim})$ is an isomorphism.* Indeed, since the open prime ideal $\mathfrak{p}$ in A corresponding to x is taken into m_x by the natural map $A \to \mathcal{O}_{\mathcal{X},x}$, therefore M is $\mathfrak{p}$-torsion. Thus M is a torsion A-module, and so there are natural isomorphisms

$$M_f \underset{2.3.1}{\xrightarrow{\sim}} M \otimes_A A_{\{f\}} \underset{2.2.1(\mathrm{iv})}{\xrightarrow{\sim}} \Gamma(\mathcal{U}_f, M^{\sim});$$

and if $x \notin \mathcal{U}_f$ then $f \in \mathfrak{p}$, so M is (f)-torsion, i.e., $M_f = 0$. If $x \in \mathcal{U}_f$ then $f \notin \mathfrak{p}$, so f maps to a unit in $\mathcal{O}_{\mathcal{X},x}$, and the natural map $M \to M_f$ is an isomorphism. The claim results.

Any open subset $\mathcal{V} \subset \mathcal{U}$ is a union of open subsets of the type $\mathcal{U}_f$, and it follows that if $x \notin \mathcal{V}$ then $M^{\sim}(\mathcal{V}) = 0$, and that if $x \in \mathcal{V}$ then the natural map $M \to M^{\sim}(\mathcal{V})$ is an isomorphism. Thus there is a natural isomorphism $i_xM|_{\mathcal{U}} \xrightarrow{\sim} M^{\sim}$. Since $M^{\sim} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{U})$ (by 2.2.1(ii) and 2.3.3), therefore $i_xM|_{\mathcal{U}} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{U})$.

(iii) $\Rightarrow$ (ii). The property of being quasi-coherent and torsion is local. If $\mathcal{V}$ is an open set not containing x then i_xM restricts to the zero sheaf and hence is in $\mathcal{A}_{\mathrm{qct}}(\mathcal{V})$. The implication follows.

(ii) $\Rightarrow$ (iii) follows easily from 2.3.4(iii); and (iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). With $\mathcal{U} = \mathrm{Spf}(A)$ as in (iv), 2.3.4(iii) gives $M^{\sim} \cong i_xM|_{\mathcal{U}} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{U})$, so by 2.3.3, M is a torsion A-module. Let $\mathfrak{p}$ be the open prime A-ideal corresponding to $x \in \mathcal{U}$. For any $f \in \mathfrak{p}$, we have $x \notin \mathcal{U}_f := \mathrm{Spf}(A_{\{f\}})$, whence, as above, there are natural isomorphisms

$$M_f \underset{2.3.1}{\xrightarrow{\sim}} M \otimes_A A_{\{f\}} \underset{2.2.1(\mathrm{iv})}{\xrightarrow{\sim}} \Gamma(\mathcal{U}_f, M^{\sim}) \xrightarrow{\sim} \Gamma(\mathcal{U}_f, i_xM) = 0,$$

so that M is (f)-torsion. As $\mathfrak{p}$ is finitely generated, M must be $\mathfrak{p}$-torsion; and since $m_x = \varinjlim_{f \notin \mathfrak{p}} \mathfrak{p}_{\{f\}} = \mathfrak{p}\mathcal{O}_{\mathcal{X},x}$ ([**10**, p. 186, proof of (7.6.17)]), M is m_x-torsion. $\square$

Let $\mathcal{X}$ be a noetherian formal scheme. For $x \in \mathcal{X}$, let $J(x)$ denote the injective hull of the residue field $k(x)$ over the local ring $\mathcal{O}_{\mathcal{X},x}$. It is easily checked that $i_xJ(x)$ is an injective object in $\mathcal{A}(\mathcal{X})$ ([**11**, page 123]). By 2.3.5, $i_xJ(x) \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$.

Recall that in a locally noetherian category a direct sum of injectives is also an injective.

Proposition 2.3.6. *Let $\mathcal{X}$ be a noetherian formal scheme.*

(i) *The categories $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$ and $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ are locally noetherian and have enough injectives.*

(ii) *The indecomposable injectives of* $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ *are those of the form* $i_x J(x)$ *defined above. In particular, any injective object* $\mathcal{I}$ *of* $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ *is a direct sum of injectives of the form* $i_x J(x)$ *and hence is injective in* $\mathcal{A}_{\mathrm{t}}(\mathfrak{X})$ *too.*

(iii) *For any* $\mathcal{M} \in \mathcal{A}_{\mathrm{t}}(\mathfrak{X}) \cup \mathcal{A}_{\mathrm{c}}(\mathfrak{X})$ *and for any injective* $\mathcal{I}$ *in* $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, *the sheaf* $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{I})$ *is flasque.*

(iv) *For any injective* $\mathcal{I}$ *in* $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, *and* $x \in \mathfrak{X}$, *the* $\mathcal{O}_{\mathfrak{X},x}$*-module* $\mathcal{I}_x$ *is injective.*

Proof. For proofs of (i), (ii) and (iii), we refer to [**27**, p. 876].

For (iv), we may assume, by (ii), that $\mathcal{I} = i_{x'} J(x')$ for suitable x'. If x' doesn't specialize to x then $\mathcal{I}_x = 0$. It remains to show that *if* x' *specializes to* x *then* $J(x')$ *is an injective* $\mathcal{O}_{\mathfrak{X},x}$*-module.* For this it suffices, with $\mathcal{O}_y := \mathcal{O}_{\mathfrak{X},y}$ $(y \in \mathfrak{Y})$, that $\widehat{\mathcal{O}_{x'}} \cong \widehat{(\mathcal{O}_x)_\wp}$ for some prime $\wp$ in $\mathcal{O}_x$, since then the injective hull $J(x')$ of the residue field $k(x')$ over $\mathcal{O}_{x'}$, being injective over the local ring $\widehat{\mathcal{O}_{x'}}$ too, is injective over $\widehat{(\mathcal{O}_x)_\wp}$ and hence over $\mathcal{O}_x$.

Let $\mathfrak{U} = \mathrm{Spf}(A)$ be an open neighborhood of x. Recalling that $\mathcal{O}_y$ is noetherian for all $y \in \mathfrak{U}$ [**10**, p. 403, (10.1.6)], one deduces from [**10**, p. 184, (7.6.9)] that if $\mathfrak{a}$ is an open A-ideal and $\mathfrak{q} = \mathfrak{q}_y$ is the open prime A-ideal corresponding to y, then $\mathcal{O}_{\mathfrak{X},y}/\mathfrak{a}\mathcal{O}_y \cong A_{\mathfrak{q}}/\mathfrak{a}A_{\mathfrak{q}}$. So if $\mathfrak{p} := \mathfrak{q}_{x'} \subset \mathfrak{q}_y$ then $\wp_y := \mathfrak{p}\mathcal{O}_y$ is prime, and for all $i \geq 0$, $(\mathcal{O}_y)_{\wp_y}/\wp_y^i(\mathcal{O}_y)_{\wp_y}$ is isomorphic to the localization of the A-module $A_{\mathfrak{q}}/\mathfrak{p}^i A_{\mathfrak{q}}$ at the prime ideal $\mathfrak{p}$, i.e., to $A_{\mathfrak{p}}/\mathfrak{p}^i A_{\mathfrak{p}}$. Hence for all y in the closure of x', $\widehat{(\mathcal{O}_y)_{\wp_y}} \cong \widehat{A_{\mathfrak{p}}}$. Taking $y = x'$ and x respectively, one gets $\widehat{\mathcal{O}_{\mathfrak{X},x'}} \cong \widehat{(\mathcal{O}_x)_{\wp_x}}$, as desired. □

2.4. Smooth maps. A homomorphism of topological rings $\phi\colon A \to B$ is *formally smooth* if ϕ is continuous and if for every discrete topological A-algebra C and every nilpotent ideal I of C, any continuous A-homomorphism $B \to C/I$ factors as $B \xrightarrow{v} C \twoheadrightarrow C/I$ with v a continuous A-homomorphism ([**8**, 19.3.1]).

A morphism of formal schemes $f\colon \mathfrak{X} \to \mathfrak{Y}$ is said to be *formally smooth* if for any morphism $Z \to \mathfrak{Y}$ where $Z = \mathrm{Spec}(C)$ is an affine scheme, and for any closed subscheme $Z_0 \subset Z$ defined by a nilpotent ideal in C, every $\mathfrak{Y}$-morphism $Z_0 \to \mathfrak{X}$ extends to a $\mathfrak{Y}$-morphism $Z \to \mathfrak{X}$ (cf. [**9**, §17.1]).

Examples 2.4.1. We recall some of the elementary properties and standard examples of formally smooth maps. These will often be used without explicit mention. For proofs cf. [**9**, §17.1] and [**8**, §19.3].

(i) An open immersion is a formally smooth map. A composition of formally smooth maps is formally smooth. Formal smoothness is preserved under base change.

(ii) A map of affine formal schemes $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is formally smooth if and only if the corresponding homomorphism of topological rings $A \to B$ is formally smooth.

(iii) For a discrete ring A, any polynomial algebra under the discrete topology is formally smooth over A.

(iv) If $\phi\colon A \to B$ is a formally smooth map of topological rings then for any multiplicative sets $S \subset A$ and $T \subset B$ such that $\phi(S) \subset T$, the induced map $S^{-1}A \to T^{-1}B$ is formally smooth.

(v) A map of topological rings $A \to B$ is formally smooth if and only if the induced map of respective completions $\widehat{A} \to \widehat{B}$ is formally smooth.

(vi) Let $\phi\colon A \to B$ be a formally smooth map of topological rings. Then for any choice of coarser topologies on A, B for which the square of any open B-ideal is open and for which ϕ remains continuous, ϕ is formally smooth under the coarser topologies too.

DEFINITION 2.4.2. A continuous homomorphism $\phi\colon A \to B$ of noetherian adic rings (resp. a map $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian formal schemes) is *smooth* if it is essentially of pseudo-finite type and formally smooth.

Recall that a homomorphism $\phi\colon R \to S$ of noetherian adic rings is *adic* if for one (hence for every) defining ideal I of R, $\phi(I)$ is a defining ideal of S.

LEMMA 2.4.3. *Let $\phi\colon A \to B$ be an essentially pseudo-finite type homomorphism of noetherian adic rings. Then ϕ factors as $A \xrightarrow{\sigma} C \xrightarrow{\pi} B$ where σ is a smooth map and π is a surjective adic homomorphism. More specifically, C can be obtained as the completion of a localization of a polynomial algebra over A—say $(A[X_1, \dots, X_n])_S$—along an ideal I that contracts to an open A-ideal.*

PROOF. Let $\mathfrak{a}, \mathfrak{b}$ be defining ideals of A, B, respectively, such that $\phi(\mathfrak{a}) \subset \mathfrak{b}$. By hypothesis there exists a finitely generated A-algebra D and a multiplicative set T in D such that $B/\mathfrak{b} \cong D_T$. Then there is a surjection from a polynomial ring $P := A[X_1, \dots, X_r]$ to D which maps $\{X_i\}$ to a set of generators of D. Let S be the inverse image of T in P. The composition $P \twoheadrightarrow D \to D_T \cong B/\mathfrak{b}$ lifts to an A-homomorphism $\psi\colon P \to B$. Any element of $\psi(S)$ maps to a unit in $D_T \cong B/\mathfrak{b}$. Since $\mathfrak{b}$, being a defining ideal of B, is in the Jacobson radical, it follows that $\psi(S)$ consists of units of B and so we obtain an induced map $\psi_S\colon P_S \to B$.

Let $\pi_0\colon Q = P_S[X_{r+1}, \dots, X_n] \to B$ be a map extending ψ_S and taking the indeterminates $X_{r+1}, \dots, X_n$ to a generating set of the B-ideal $\mathfrak{b}$. Let I be the Q-ideal generated by $\mathfrak{a}$ and $X_{r+1}, \dots, X_n$. Then π_0 is continuous for the I-adic topology on Q and hence extends to a map π, clearly adic, from the I-adic completion $C := \widehat{Q}$ to B. Since $\widehat{I}B = \mathfrak{b}$, therefore B is an $\widehat{I}$-adically separated C-module and C surjects onto $B/(\widehat{I}B)$. It follows from [**19**, Theorem 8.4] that π is surjective. $\square$

COROLLARY 2.4.4. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is an essentially pseudo-finite type map of formal schemes then any neighborhood of a point $x \in \mathcal{X}$ contains an affine open neighborhood $\mathcal{U}$ such that $f|_{\mathcal{U}} = hi$ where for some formal scheme $\mathcal{Z}$, $i\colon \mathcal{U} \to \mathcal{Z}$ is a closed immersion and $h\colon \mathcal{Z} \to \mathcal{Y}$ is smooth.*

Let $A \xrightarrow{\phi} B$ be a continuous homomorphism of noetherian adic rings and let $\mathcal{U} := \mathrm{Spf}(B) \xrightarrow{f} \mathcal{V} := \mathrm{Spf}(A)$ be the induced map of formal schemes. Let $x \in \mathcal{U}$ and $y := f(x)$. Let $k(x)$ and $k(y)$ denote the residue fields of the local rings $\mathcal{O}_{\mathcal{U},x}$ and $\mathcal{O}_{\mathcal{V},y}$ respectively. We regard these local rings as being topologized by the powers of the stalks of defining ideals on $\mathcal{U}$ and $\mathcal{V}$ respectively. Thus if $\mathfrak{a}, \mathfrak{b}$ are defining ideals of A, B then $\mathfrak{a}\mathcal{O}_{\mathcal{U},x}$, $\mathfrak{b}\mathcal{O}_{\mathcal{V},y}$ are defining ideals of $\mathcal{O}_{\mathcal{U},x}$, $\mathcal{O}_{\mathcal{V},y}$, respectively.

PROPOSITION 2.4.5. *In the preceding situation we have*:

(i) *If ϕ is essentially of pseudo-finite type then $k(x)$ is a finitely generated field extension of $k(y)$.*

(ii) *If ϕ is formally smooth, then ϕ is flat. Furthermore, the induced map of topological rings $\mathcal{O}_{\mathcal{V},y} \to \mathcal{O}_{\mathcal{U},x}$ is also formally smooth and flat. In particular, $f\colon \mathcal{U} \to \mathcal{V}$ is a flat map of formal schemes.*

Proof. (i). Let $\mathfrak{b}$ be a defining ideal in B and $\mathfrak{a} \subset \phi^{-1}\mathfrak{b}$ a defining ideal in A. Then $A/\mathfrak{a} \to B/\mathfrak{b}$ is essentially of finite type. As in the proof of 2.3.6(iv), $\mathcal{O}_{\mathcal{V},y}/\mathfrak{a}\mathcal{O}_{\mathcal{V},y}$ and $\mathcal{O}_{\mathcal{U},x}/\mathfrak{b}\mathcal{O}_{\mathcal{U},x}$ are localizations of $A/\mathfrak{a}$ and $B/\mathfrak{b}$ respectively. Therefore the induced map $k(x) \to k(y)$ is essentially of finite type, whence the result.

(ii). Let $\mathfrak{n}$ be a maximal B-ideal and $\mathfrak{m} = \phi^{-1}\mathfrak{n}$. Let $\mathfrak{b}$ be a defining ideal in B and $\mathfrak{a} \subset \phi^{-1}\mathfrak{b}$ a defining ideal in A. Then the induced map $A_\mathfrak{m} \to B_\mathfrak{n}$ is formally smooth under the $\mathfrak{a}$-adic and $\mathfrak{b}$-adic topologies respectively. The map $A_\mathfrak{m} \to B_\mathfrak{n}$ is also continuous for the coarser $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies respectively. Therefore $A_\mathfrak{m} \to B_\mathfrak{n}$ is formally smooth for the coarser topologies too. By [**8**, 19.7.1], $B_\mathfrak{n}$ is flat over $A_\mathfrak{m}$, hence over A. This being so for any $\mathfrak{n}$, B is flat over A.

For the final part, let $\mathfrak{p} \subset B$ (resp. $\mathfrak{q} \subset A$) be the open prime corresponding to the point x (resp. y). As in the proof of 2.3.6(iv), the completion of $\mathcal{O}_{\mathcal{U},x}$ (resp. $\mathcal{O}_{\mathcal{V},y}$) along the ideal $\mathfrak{b}\mathcal{O}_{\mathcal{U},x}$ (resp. $\mathfrak{a}\mathcal{O}_{\mathcal{V},y}$) is isomorphic to the completion of $B_\mathfrak{p}$ (resp. $A_\mathfrak{q}$) along the ideal $\mathfrak{b}_\mathfrak{p}$ (resp, $\mathfrak{a}_\mathfrak{q}$). Therefore [$A \to B$ formally smooth] $\Longrightarrow$ [$A_\mathfrak{q} \to B_\mathfrak{p}$ formally smooth] $\Longrightarrow$ [$\widehat{A_\mathfrak{q}} \to \widehat{B_\mathfrak{p}}$ formally smooth] $\Longrightarrow$ [$\mathcal{O}_{\mathcal{V},y} \to \mathcal{O}_{\mathcal{U},x}$ formally smooth] ([**8**, 19.3.5, (iv)], [**8**, 19.3.6]). Again, by [**8**, 19.7.1], $\mathcal{O}_{\mathcal{V},y} \to \mathcal{O}_{\mathcal{U},x}$ is flat. □

2.5. Differentials on topological rings. We now describe various elementary properties of the module of relative differentials for a map of topological rings, and its behavior under smooth maps. Let $A \to B$ be a continuous homomorphism of adic rings. Let $\mathfrak{b}$ be any defining ideal in B. Let $\Omega^1_{B/A}$ be the relative B-module of differentials and $\Omega^m_{B/A}$ its m-th exterior power ($m \ge 0$), $\mathfrak{b}$-adically topologized (cf. [**8**, 20.4.5]). Set $B_i := B/\mathfrak{b}^{i+1}$, and

$$\widehat{\Omega^m_{B/A}} := \varprojlim_i \, (\Omega^m_{B/A} \otimes_B B_i),$$

the $\mathfrak{b}$-adic completion of $\Omega^m_{B/A}$. Also, set

$$\Omega^{\mathrm{sep}}_{B/A} := \Omega^1_{B/A}/(\cap_i \mathfrak{b}^i \Omega^1_{B/A}),$$

the universal separated module of differentials of B/A. These definitions do not depend on the choice of $\mathfrak{b}$. Indeed, the canonical A-derivation $d_{B/A}\colon B \to \Omega^1_{B/A}$ induces derivations $d^{\mathrm{sep}}_{B/A}\colon B \to \Omega^{\mathrm{sep}}_{B/A}$ and $\widehat{d}_{B/A}\colon B \to \widehat{\Omega^1_{B/A}}$ which are universal for continuous A-derivations of B into separated (resp. complete separated) B-modules.

Recall that for any topological B-module N, composition with $d_{B/A}$ gives an isomorphism $\mathrm{Hom}^{\mathrm{c}}_B(\Omega^1_{B/A}, N) \xrightarrow{\ \sim\ } \mathrm{Der}^{\mathrm{c}}_A(B, N)$, where the superscript "c" signifies, respectively, *continuous* homomorphisms and derivations (see [**8**, 20.4.8.2]). Let $\mathfrak{b}$ be a defining ideal of B. If the topology on N is $\mathfrak{b}$-adic and N is separated (e.g., if N is finitely generated) then we also have the following relations.

$$\mathrm{Der}^{\mathrm{c}}_A(B, N) = \mathrm{Der}_A(B, N),$$

$$\mathrm{Hom}^{\mathrm{c}}_B(\Omega^1_{B/A}, N) \xleftarrow{\ \sim\ } \mathrm{Hom}^{\mathrm{c}}_B(\Omega^{\mathrm{sep}}_{B/A}, N) = \mathrm{Hom}_B(\Omega^{\mathrm{sep}}_{B/A}, N) \xrightarrow{\ \sim\ } \mathrm{Hom}_B(\Omega^1_{B/A}, N).$$

All these relations are easily verified. For example, the first one holds because for any $\delta \in \mathrm{Der}_A(B, N)$ we have $\delta(\mathfrak{b}^{i+1}B) \subset \mathfrak{b}^i N$.

Proposition 2.5.1. *Let $f\colon A \to B$ be an essentially pseudo-finite type map of noetherian adic rings. Let $\mathfrak{b}$ be a defining B-ideal and $B_i := B/\mathfrak{b}^{i+1}$. Let $m \geq 0$. Then:*

(i) *The B-module $\Omega^{\mathrm{sep}}_{B/A}$ is finitely generated and complete; and there is a natural isomorphism $\Omega^{\mathrm{sep}}_{B/A} \xrightarrow{\,\sim\,} \widehat{\Omega^1_{B/A}}$.*

(ii) *There are natural isomorphisms*

$$\bigwedge_B^m \widehat{\Omega^1_{B/A}} \xrightarrow{\,\sim\,} \widehat{\Omega^m_{B/A}} \xrightarrow{\,\sim\,} \varprojlim_i \Omega^m_{B_i/A}.$$

In particular, $\widehat{\Omega^m_{B/A}}$ is a finitely generated B-module.

(iii) *If f is smooth then $\widehat{\Omega^m_{B/A}}$ is a projective B-module.*

(iv) *If $\mathfrak{b}$ is a defining ideal of B, $\mathfrak{b}' \subset \mathfrak{b}$, and $B' := B$ with $\mathfrak{b}'$-adic topology (also an adic ring) is essentially of pseudo-finite type over A, then the natural map is an isomorphism*

$$\widehat{\Omega^1_{B'/A}} \xrightarrow{\,\sim\,} \widehat{\Omega^1_{B/A}}.$$

Proof. B_0 is essentially of finite type over A, so $\Omega^1_{B_0/A}$ is a finitely generated B_0-module and hence a finitely generated B-module. Then the instance $i = 0$ of the natural exact sequences

$$\mathfrak{b}^{i+1}/\mathfrak{b}^{i+2} \underset{\text{via } d}{\longrightarrow} \Omega^1_{B/A} \otimes_B B_i \longrightarrow \Omega^1_{B_i/A} \longrightarrow 0 \qquad (i \geq 0) \tag{4}$$

shows that $\Omega^1_{B/A} \otimes_B B_0$ is a finitely generated B-module, so that the B-module $\Omega^{\mathrm{sep}}_{B/A} \otimes_B B_0 \cong \Omega^1_{B/A} \otimes_B B_0$ is finitely generated. By [**19**, Thm. 8.4] it follows that the B-module $\Omega^{\mathrm{sep}}_{B/A}$ is finitely generated, hence complete; and so $\Omega^{\mathrm{sep}}_{B/A} = \widehat{\Omega^1_{B/A}}$.

Furthermore, the exact sequence (4) gives that the kernel k_i of the natural surjection $\Omega^1_{B/A} \twoheadrightarrow \Omega^1_{B_i/A}$ satisfies

$$\mathfrak{b}^{i+1}\Omega^1_{B/A} \subset k_i = \mathfrak{b}^{i+1}\Omega^1_{B/A} + d(\mathfrak{b}^{i+1}) \subset \mathfrak{b}^i\Omega^1_{B/A},$$

giving rise to a sequence

$$\cdots \longrightarrow \Omega^1_{B_{i+1}/A} \longrightarrow \Omega^1_{B/A} \otimes_B B_i \longrightarrow \Omega^1_{B_i/A} \longrightarrow \Omega^1_{B/A} \otimes_B B_{i-1} \longrightarrow \cdots,$$

from which follows, upon application of $\bigwedge_B^m$, a natural sequence

$$\cdots \longrightarrow \Omega^m_{B_{i+1}/A} \longrightarrow \Omega^m_{B/A} \otimes_B B_i \longrightarrow \Omega^m_{B_i/A} \longrightarrow \Omega^m_{B/A} \otimes_B B_{i-1} \longrightarrow \cdots,$$

and hence a natural isomorphism $\widehat{\Omega^m_{B/A}} \xrightarrow{\,\sim\,} \varprojlim_i \Omega^m_{B_i/A}$.

By part (i), $\bigwedge^m \widehat{\Omega^1_{B/A}}$ is a finitely generated B-module and hence is complete. With $B_i := B/\mathfrak{b}^{i+1}$ there are natural isomorphisms

$$\Big(\bigwedge^m \widehat{\Omega^1_{B/A}}\Big) \otimes_B B_i \cong \bigwedge^m \big(\widehat{\Omega^1_{B/A}} \otimes_B B_i\big) \cong \bigwedge^m \big(\Omega^1_{B/A} \otimes_B B_i\big) \cong \Big(\bigwedge^m \Omega^1_{B/A}\Big) \otimes_B B_i.$$

Taking inverse limit over i we conclude that $\bigwedge^m \widehat{\Omega^1_{B/A}} \cong \widehat{\Omega^m_{B/A}}$, proving (ii).

Now suppose f is smooth. By [**8**, Cor. 20.4.10], $\widehat{\Omega^1_{B/A}} \otimes_B B_i$ is a projective B_i-module for each i. Hence by [**19**, Thm. 22.1], $\widehat{\Omega^1_{B/A}}$ is a projective B-module. Taking exterior powers and using (ii), we deduce (iii).

The natural map in (iv) is the obvious one from the completion of a module to the completion of the same module with coarser topology. That it is an isomorphism follows from the fact that every finitely-generated B-module is $\mathfrak{b}$- and $\mathfrak{b}'$-complete and separated, whence the canonical derivations $d\colon B \to \widehat{\Omega^1_{B/A}}$ and $d'\colon B \to \widehat{\Omega^1_{B'/A}}$ are both universal for A-derivations of B into finitely-generated B-modules. $\square$

From now on, for any continuous homomorphism $A \to B$ that is essentially of pseudo-finite type we shall denote $\widehat{\Omega^m_{B/A}}$ by $\widehat{\Omega}^m_{B/A}$.

Lemma 2.5.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be essentially pseudo-finite-type homomorphisms of noetherian adic rings. Then there is a natural exact sequence of continuous C-module maps*

$$\widehat{\Omega}^1_{B/A} \otimes_B C \to \widehat{\Omega}^1_{C/A} \to \widehat{\Omega}^1_{C/B} \to 0.$$

If, moreover, g is smooth (2.4.2) *then this sequence is part of a split exact sequence*

$$0 \to \widehat{\Omega}^1_{B/A} \otimes_B C \to \widehat{\Omega}^1_{C/A} \to \widehat{\Omega}^1_{C/B} \to 0.$$

Proof. The map $\widehat{\Omega}^1_{B/A} \otimes_B C \to \widehat{\Omega}^1_{C/A}$ corresponds to the continuous A-derivation $B \to C \xrightarrow{\hat{d}_{C/A}} \widehat{\Omega}^1_{C/A}$. (Note here that by 2.5.1(i), $\widehat{\Omega}^1_{B/A}$ is a finitely generated B-module, so that $\widehat{\Omega}^1_{B/A} \widehat{\otimes}_B C \cong \widehat{\Omega}^1_{B/A} \otimes_B C$ (see [**10**, p. 189, 7.7.9]); and that $\widehat{\Omega}^1_{C/A}$, being finitely generated over C, is a complete separated B-module.) The map $\widehat{\Omega}^1_{C/A} \to \widehat{\Omega}^1_{C/B}$ corresponds to the continuous A-derivation $\hat{d}_{C/B}$. Exactness follows from that of the dual sequence—for any separated complete C-module N:

$$\mathrm{Der}^{\mathrm{c}}_A(B,N) \leftarrow \mathrm{Der}^{\mathrm{c}}_A(C,N) \leftarrow \mathrm{Der}^{\mathrm{c}}_B(C,N) \leftarrow 0.$$

(See also [**8**, p. 152, 20.7.17.3].) For smooth g one can then apply [**8**, 20.7.18], in view of *ibid.,* p. 114, Definition 19.9.1. (In fact, arguing as in the proof of *loc. cit.,* one sees that $\mathrm{Der}^{\mathrm{c}}_A(C,N) \to \mathrm{Der}^{\mathrm{c}}_A(B,N)$ is surjective.) $\square$

Lemma 2.5.3. *Let A, B, C, be noetherian adic rings, let $f\colon A \to B$ be a homomorphism essentially of pseudo-finite type, and let $\pi\colon B \to C$ be a continuous surjection, with kernel I. Suppose that the topological quotient algebra B/I is smooth over A. Then there is a split exact sequence*

$$0 \to I/I^2 \to \widehat{\Omega}^1_{B/A} \otimes_B C \to \widehat{\Omega}^1_{C/A} \to 0.$$

Proof. In view of 2.5.1(iv), one may assume that $C = B/I$. Let $\mathfrak{b}$ be a defining B-ideal. Consider the diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & (\cap_i\, \mathfrak{b}^i\Omega^1_{B/A}) \otimes_B C & \longrightarrow & \cap_i\, \mathfrak{b}^i\Omega^1_{C/A} \\
\downarrow & & \downarrow & & \downarrow \\
I/I^2 & \longrightarrow & \Omega^1_{B/A} \otimes_B C & \longrightarrow & \Omega^1_{C/A}
\end{array}$$

wherein the bottom row consists of usual natural maps, the top row is induced by the bottom row, and the vertical maps are the natural ones. The diagram clearly commutes and so the cokernels of the vertical maps lie in a sequence

$$E^\bullet : \qquad 0 \to I/I^2 \to \Omega^{\mathrm{sep}}_{B/A} \otimes_B C \to \Omega^{\mathrm{sep}}_{C/A} \to 0.$$

By 2.5.1(i) every module in $E^\bullet$ is finitely generated over C, so for $E^\bullet$ to be split exact it suffices that for any finitely generated C-module N the induced sequence $\mathrm{Hom}_C(E^\bullet, N)$ is exact. Rewriting $\mathrm{Hom}_C(E^\bullet, N)$ as

$$0 \to \mathrm{Der}_A(C, N) \to \mathrm{Der}_A(B, N) \xrightarrow{u} \mathrm{Hom}_C(I/I^2, N) \to 0$$

one sees that the only nontrivial thing to show is surjectivity of u.

Since the kernel I/I^2 of the natural surjection $\overline{\pi}\colon B/I^2 \to C$ is nilpotent and closed, and C is smooth over A, therefore the identity map of C lifts to an A-homomorphism $l\colon C \to B/I^2$ [**8**, p. 84, 19.3.10]. Then

$$\sigma := 1 - l\overline{\pi}\colon B/I^2 \to I/I^2$$

is an A-derivation: for all $x, y \in B/I^2$,

$$\sigma(xy) = xy - l\overline{\pi}(xy) = xy - l\overline{\pi}(x)l\overline{\pi}(y) = xy - (x-\sigma(x))(y-\sigma(y)) = x\sigma(y) + y\sigma(x).$$

So for any $\delta \in \mathrm{Hom}_C(I/I^2, N)$, $\delta\sigma \in \mathrm{Der}_A(B/I^2, N) = \mathrm{Der}_A(B, N)$ and $u(\delta\sigma) = \delta$. This proves surjectivity of u, and hence split-exactness of $E^\bullet$. □

Example 2.5.4. Let B be a noetherian adic ring, S a multiplicative set in B and I an open ideal in B. Set $C := B\{S^{-1}\}$, the completion of B_S along I_S. Then the induced map $B \to C$ is smooth.

Any continuous B-derivation from C into a topological C-module vanishes on B_S, hence, by continuity, on C. Therefore, $\widehat{\Omega}^1_{C/B} = 0$.

Any continuous A-derivation from B into a complete separated C-module extends uniquely to C. Hence (or by 2.5.2) there is a natural isomorphism

$$\widehat{\Omega}^1_{B/A} \mathbin{\widehat{\otimes}}_B B\{S^{-1}\} \xrightarrow{\sim} \widehat{\Omega}^1_{B\{S^{-1}\}/A}.$$

Example 2.5.5. Let A be a noetherian adic ring and $P = A[X_1, \dots, X_r]$ a polynomial ring over A. Let I be an ideal in P such that $I \cap A$ is open in A, and let C be the completion of P along I. Then the natural map $A \to C$ is smooth. As any A-derivation of P into an I-adically complete separated C-module extends uniquely to C, one sees that the C-module $\widehat{\Omega}^1_{C/A}$ is free, with basis $dX_1, \dots, dX_r$.

Proposition 2.5.6. *Let $A \to B$ and $A \to C$ be continuous maps of adic rings. Set $D := B \mathbin{\widehat{\otimes}}_A C$. Then there exists a natural isomorphism $\widehat{\Omega}^1_{B/A} \mathbin{\widehat{\otimes}}_B D \xrightarrow{\sim} \widehat{\Omega}^1_{D/C}$.*

Proof. (i) Via the universal properties of $\widehat{\Omega}^1$ and of extension of scalars (from B to D), the Proposition amounts to the statement that for any complete separated D-module L, "restriction" induces a bijection $\mathrm{Der}^c_C(D, L) \xrightarrow{\sim} \mathrm{Der}^c_A(B, L)$. But if we give the D-algebra $\mathcal{D} := D \oplus L$ (where $L^2{=}0$) the product topology, under which it is complete and separated, then the standard correspondence between derivations into L and ring homomorphisms into $\mathcal{D}$ [**8**, p. 118, 20.1.5] respects continuity, and so transforms the statement into bijectivity of the restriction map from continuous C-homomorphisms $D \to \mathcal{D}$ to continuous A-homomorphisms $B \to \mathcal{D}$, which is easily seen to hold, by the universal property of complete tensor products. □

Remark. When A and B are noetherian and $A \to B$ is essentially of pseudo-finite type, one can replace the $\widehat{\otimes}$ in 2.5.6 by $\otimes$, see proof of 2.5.2. Moreover, in view of 2.5.1(ii), one can replace $\widehat{\Omega}^1$ by $\widehat{\Omega}^m$ for any $m \geq 0$.

2.6. Differentials on formal schemes. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian formal schemes. Suppose $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ and $\mathcal{J} \subset \mathcal{O}_{\mathcal{Y}}$ are defining ideals such that $\mathcal{J}\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$. There result morphisms of ordinary schemes

$$X_n := (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1}) \xrightarrow{f_n} (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\mathcal{J}^{n+1}) =: Y_n \qquad (n > 0).$$

Let $j_n\colon X_n \hookrightarrow \mathcal{X}$ be the canonical closed immersion. For $(m \geq 0)$ let $\Omega^m_{X_n/Y_n}$ be the m-th exterior power of the sheaf of relative differentials on X_n, and set $\widehat{\Omega}^m_f = \widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}} := \varprojlim_n j_{n*}\Omega^m_{X_n/Y_n}$. This $\widehat{\Omega}^m_f$ is independent of the choice of $\mathcal{I}$, $\mathcal{J}$. For every n there is a natural sheaf homomorphism $\mathcal{O}_{X_n} \to \Omega^1_{X_n/Y_n}$, and hence applying j_{n*} and taking inverse limits results in a natural sheaf homomorphism $\hat{d}_{\mathcal{X}/\mathcal{Y}} = \hat{d}_f\colon \mathcal{O}_{\mathcal{X}} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}}$.

Let $\mathrm{Spf}(B) = \mathcal{U} \subset \mathcal{X}$ and $\mathrm{Spf}(A) = \mathcal{V} \subset \mathcal{Y}$ be affine open subschemes with $f(\mathcal{U}) \subset \mathcal{V}$. Let $\mathrm{Spec}(A_n) = U_n \subset \mathcal{U}$ and $\mathrm{Spec}(B_n) = V_n \subset \mathcal{V}$ be defined as above. Then for each $m \geq 0$ there are natural isomorphisms

$$\begin{aligned}
\Gamma(\mathcal{U}, \widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}}) &= \Gamma(\mathcal{U}, \varprojlim_n j_{n*}\Omega^m_{X_n/Y_n}) \cong \varprojlim_n \Gamma(\mathcal{U}, j_{n*}\Omega^m_{X_n/Y_n}) \\
&= \varprojlim_n \Gamma(U_n, \Omega^m_{X_n/Y_n}) \cong \varprojlim_n \Gamma(U_n, \Omega^m_{U_n/V_n}) \\
&\cong \varprojlim_n \Omega^m_{B_n/A_n} \cong \varprojlim_n \Omega^m_{B_n/A} \underset{2.5.1(\mathrm{ii})}{\cong} \widehat{\Omega}^m_{B/A}.
\end{aligned} \tag{5}$$
[3]

Proposition 2.6.1. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an essentially pseudo-finite type map of noetherian formal schemes. Then for $m \geq 0$, the natural map is an isomorphism*

$$\bigwedge^m_{\mathcal{O}_{\mathcal{X}}} \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\ \sim\ } \varprojlim_n \bigwedge^m_{\mathcal{O}_{\mathcal{X}}} j_{n*}\Omega^1_{X_n/Y_n} =: \widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}},$$

and $\widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module. Moreover, if $\mathcal{U} = \mathrm{Spf}(B)$ and $\mathcal{V} = \mathrm{Spf}(A)$ are as above, and if the induced map $A \to B$ is essentially of pseudo-finite type, then, with $\sim = \sim_B$, there are natural isomorphisms

$$\big(\widehat{\Omega}^m_{B/A}\big)^{\sim} \xrightarrow{\ \sim\ } \widehat{\Omega}^m_{\mathcal{U}/\mathcal{V}} \xrightarrow{\ \sim\ } \widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}}\big|_{\mathcal{U}}.$$

Furthermore, if f is smooth then f is flat and $\widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}}$ is locally free of finite rank.

Proof. All the assertions are local, so we may assume that $\mathcal{X} = \mathcal{U}$ and $\mathcal{Y} = \mathcal{V}$. For any affine open subset $\mathcal{U}_b = \mathrm{Spf}(B_{\{b\}}) \subset \mathcal{U}$ we have the natural isomorphisms

$$\Gamma(\mathcal{U}_b, \widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}}) \underset{(5)}{\cong} \widehat{\Omega}^m_{B_{\{b\}}/A} \cong \widehat{\Omega}^m_{B/A} \otimes_B B_{\{b\}} \cong \Gamma(\mathcal{U}_b, (\widehat{\Omega}^m_{B/A})^{\sim}),$$

where the second isomorphism comes from 2.5.4, and the third from 2.2.1(iv). It follows that the map corresponding to the isomorphism $\widehat{\Omega}^m_{B/A} \cong \Gamma(\mathcal{U}, \widehat{\Omega}^m_{\mathcal{U}/\mathcal{V}})$ of (5) is itself an isomorphism $(\widehat{\Omega}^m_{B/A})^{\sim} \xrightarrow{\ \sim\ } \widehat{\Omega}^m_{\mathcal{U}/\mathcal{V}}$. Since $\widehat{\Omega}^m_{B/A}$ is a finitely-generated B-module (see 2.5.1(ii)), $\widehat{\Omega}^m_{\mathcal{U}/\mathcal{V}}$ is coherent; and the isomorphism $\bigwedge^m \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\ \sim\ } \widehat{\Omega}^m_{\mathcal{X}/\mathcal{Y}}$ results from 2.5.1(ii) and 2.2.3(iv).

The assertions concerning smooth maps follow from 2.4.5(ii) and 2.5.1(iii). □

[3]The hypothesis in 2.5.1 that B be essentially of pseudo-finite type over A is not used in the proof of this last isomorphism.

DEFINITION 2.6.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of noetherian formal schemes. The *relative dimension of* f is the function—constant on connected components of $\mathcal{X}$—taking $x \in \mathcal{X}$ to the rank of $\widehat{\Omega}^1_f$ at x.

When the relative dimension is constant on all of $\mathcal{X}$, we may identify it with its value (a nonnegative integer).

EXAMPLE 2.6.3. An open immersion is smooth of relative dimension 0 (2.5.4). The map $\mathrm{Spf}(C) \to \mathrm{Spf}(A)$ obtained from 2.5.5 is smooth of relative dimension r.

DEFINITION 2.6.4. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of noetherian formal schemes. With d_i the relative dimension of f on the connected component $\mathcal{X}_i$ of $\mathcal{X}$, we denote by ω_f the invertible $\mathcal{O}_{\mathcal{X}}$-module whose restriction to $\mathcal{X}_i$ is $\widehat{\Omega}^{d_i}_{\mathcal{X}/\mathcal{Y}}$.

PROPOSITION 2.6.5. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be smooth maps of relative dimensions d and e respectively. Then gf is smooth of relative dimension $d+e$ and there is a canonical isomorphism*

$$f^*\omega_g \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_f \xrightarrow{\ \sim\ } \omega_{gf}.$$

PROOF. The question is local on $\mathcal{X}$, so we may assume that $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are connected, and that d and e are integers.

Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be defining ideals in $\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Z}}$ respectively such that $\mathcal{J}\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$ and $\mathcal{K}\mathcal{O}_{\mathcal{Y}} \subset \mathcal{J}$. Reducing modulo the n-th powers of these defining ideals we get maps of schemes $X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n$ and hence a sequence

$$0 \to f_n^*\Omega^1_{Y_n/Z_n} \to \Omega^1_{X_n/Z_n} \to \Omega^1_{X_n/Y_n} \to 0.$$

Let $j_n\colon X_n \to \mathcal{X}$ be the canonical immersion. Applying j_{n*} and taking inverse limits we obtain a sequence

$$0 \to \varprojlim_n\, j_{n*}f_n^*\Omega^1_{Y_n/Z_n} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \to 0.$$

Let $i_n\colon Y_n \to \mathcal{Y}$ be the canonical immersion. There are natural maps

$$f^*\varprojlim_n\, i_{n*} \to \varprojlim_n\, f^*i_{n*} \to \varprojlim_n\, j_{n*}f_n^*.$$

Hence there is a natural map

$$f^*\widehat{\Omega}^1_{\mathcal{Y}/\mathcal{Z}} = f^*\varprojlim_n\, i_{n*}\Omega^1_{Y_n/Z_n} \to \varprojlim_n\, j_{n*}f_n^*\Omega^1_{Y_n/Z_n},$$

and so we have a natural sequence

$$0 \to f^*\widehat{\Omega}^1_{\mathcal{Y}/\mathcal{Z}} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \to 0.$$

One checks, using 2.2.4, that if $\mathcal{X} = \mathrm{Spf}(C)$, $\mathcal{Y} = \mathrm{Spf}(B)$, and $\mathcal{Z} = \mathrm{Spf}(A)$ are affine and if the induced maps $A \to B \to C$ are essentially of pseudo-finite type, then this sequence is the same as the one obtained by sheafifying, via $\sim_C$, a split exact sequence as in 2.5.2. Exactness being a local property, we have therefore constructed, in the general case, an exact sequence of locally free sheaves (see 2.6.1). It follows that the rank of $\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}}$ is $d+e$, and that there is a canonical isomorphism

$$\bigwedge_{\mathcal{X}}^{d+e} \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}} \cong \bigwedge_{\mathcal{X}}^{e} f^*\widehat{\Omega}^1_{\mathcal{Y}/\mathcal{Z}} \otimes \bigwedge_{\mathcal{X}}^{d} \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}}.$$

Since f^* commutes with exterior powers, the Lemma results. $\square$

PROPOSITION 2.6.6. *Let $f\colon \mathcal{X} \to \mathcal{Z}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be formal-scheme maps, with f essentially of pseudo-finite type, so that the projection $q\colon \mathcal{W} := \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ is also essentially of pseudo-finite type* (*cf. proof of* 2.1.3):

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = \mathcal{W} & \xrightarrow{\;q\;} & \mathcal{Y} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow[\;f\;]{} & \mathcal{Z} \end{array}$$

Then for the projection $p\colon \mathcal{W} \to \mathcal{X}$ there is a natural isomorphism

$$p^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}} \xrightarrow{\;\sim\;} \widehat{\Omega}^1_{\mathcal{W}/\mathcal{Y}}.$$

PROOF. This is just a globalization of the noetherian case of 2.5.6, and can be proved similarly (or reduced, via pasting of local maps, to *loc. cit.*, see 2.2.4). Indeed, since both $p^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}}$ and $\widehat{\Omega}^1_{\mathcal{W}/\mathcal{Y}}$ are coherent $\mathcal{O}_{\mathcal{W}}$-modules (see 2.6.1), it suffices to find a natural isomorphism, functorial in the coherent $\mathcal{O}_{\mathcal{W}}$-module $\mathcal{N}$,

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{W}}}(\widehat{\Omega}^1_{\mathcal{W}/\mathcal{Y}}, \mathcal{N}) \xrightarrow{\;\sim\;} \mathrm{Hom}_{\mathcal{O}_{\mathcal{W}}}(p^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}}, \mathcal{N}),$$

which can be done, as in the proof of 2.5.6, via restriction of (sheafified) derivations. Details are left to the reader. $\square$

COROLLARY 2.6.7. *With hypotheses as in* 2.6.6, *assume further that f is smooth of constant relative dimension $d \in \mathbb{N}$. Then q is also smooth of relative dimension d. Furthermore there exists a natural isomorphism $p^*\omega_f \xrightarrow{\sim} \omega_q$.*

PROPOSITION 2.6.8. *Let $\mathcal{Z} \xrightarrow{i} \mathcal{X} \xrightarrow{h} \mathcal{Y}$ be maps of formal schemes where i is a closed immersion, h is essentially of pseudo-finite type and hi is smooth. Let $\mathcal{I}$ be the coherent ideal in $\mathcal{O}_{\mathcal{X}}$ corresponding to i. Then there is an exact sequence of $\mathcal{O}_{\mathcal{Z}}$-modules*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2\big|_{\mathcal{Z}} \longrightarrow i^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \longrightarrow \widehat{\Omega}^1_{\mathcal{Z}/\mathcal{Y}} \longrightarrow 0.$$

PROOF. The natural map $i^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \to \widehat{\Omega}^1_{\mathcal{Z}/\mathcal{Y}}$ is defined in the same manner as the map $f^*\widehat{\Omega}^1_{\mathcal{Y}/\mathcal{Z}} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Z}}$ is defined in the proof of 2.6.5. Consider the natural sheaf homomorphism

$$\psi\colon \mathcal{I} \to \mathcal{O}_{\mathcal{X}} \xrightarrow{\hat{d}_{\mathcal{X}/\mathcal{Y}}} \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \to \widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}/\mathcal{I}.$$

Let $\mathcal{U} = \mathrm{Spf}(B) \subset \mathcal{X}$, $\mathcal{V} = \mathrm{Spf}(A) \subset \mathcal{Y}$ be affine open subsets such that $h(\mathcal{U}) \subset \mathcal{V}$ and the induced map $A \to B$ is essentially of pseudo-finite type. Let $I = \mathcal{I}(\mathcal{U})$, so that via 2.2.1(iii) (see also Remark (a)), $I^{\sim_B} = \mathcal{I}$. By 2.2.3(iii) $I^2 = \mathcal{I}^2(\mathcal{U})$, and by (i) and (iii) of 2.2.1, $I/I^2 = (\mathcal{I}/\mathcal{I}^2)(\mathcal{U})$. Using 2.6.1 and 2.2.3(ii), we see that ψ induces over $\mathcal{U}$, the obvious natural map $\psi(\mathcal{U})\colon I \to \widehat{\Omega}^1_{B/A} \twoheadrightarrow \widehat{\Omega}^1_{B/A} \otimes_B B/I$. Now $\psi(\mathcal{U})$ sends I^2 to 0 and the induced map $I/I^2 \to \widehat{\Omega}^1_{B/A} \otimes_B B/I$ is B/I-linear. Since $\mathcal{U}, \mathcal{V}$ can be chosen to be arbitrarily small, it follows that ψ sends $\mathcal{I}^2$ to 0 and that the induced sheaf homomorphism $\mathcal{I}/\mathcal{I}^2\big|_{\mathcal{Z}} \to i^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}}$ is $\mathcal{O}_{\mathcal{Z}}$-linear. We therefore have a sequence of $\mathcal{O}_{\mathcal{Z}}$-modules

$$(*) \qquad 0 \longrightarrow \mathcal{I}/\mathcal{I}^2\big|_{\mathcal{Z}} \longrightarrow i^*\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}} \longrightarrow \widehat{\Omega}^1_{\mathcal{Z}/\mathcal{Y}} \longrightarrow 0.$$

Exactness of $(*)$ is a local property. If $\mathcal{Z} = \mathrm{Spf}(C)$, $\mathcal{X} = \mathrm{Spf}(B)$, $\mathcal{Y} = \mathrm{Spf}(A)$ are affine, then the sequence $(*)$ is the same as the one obtained by sheafifying, via $\sim_C$, the exact sequence in 2.5.3. Therefore $(*)$ is also exact. □

For the next result we use the following fact. If $\mathcal{W} = \mathrm{Spf}(C)$ is an affine formal scheme, then for any $w \in \mathcal{W}$, the complete local ring $D := (\mathcal{O}_{\mathcal{W},w}, m_w)\widehat{\ }$ is isomorphic to $(C_{\mathfrak{p}}, \mathfrak{p}C_{\mathfrak{p}})\widehat{\ }$ where $\mathfrak{p}$ is the open prime in C corresponding to w (see proof of 2.3.6(iv)). Therefore the natural map $C \to D$ is smooth of relative dimension 0 (see 2.5.4).

Proposition 2.6.9. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an essentially-pseudo-finite-type map of formal schemes. Let $x \in \mathcal{X}$, set $B := \mathcal{O}_{\mathcal{X},x}$, $A := \mathcal{O}_{\mathcal{Y},f(x)}$, and let $\widehat{B}$, $\widehat{A}$ be the completions of B, A along their respective maximal ideals. Then, with $\widehat{\Omega}^m_{f,x}$ the stalk of $\widehat{\Omega}^m_f$ at x, there are natural isomorphisms*

$$\widehat{\Omega}^m_{f,x} \otimes_B \widehat{B} \cong \widehat{\Omega}^m_{\widehat{B}/\widehat{A}} \qquad (m \ge 0).$$

Proof. Let $\mathcal{U} = \mathrm{Spf}(S), \mathcal{V} = \mathrm{Spf}(R)$ be affine neighborhoods of x, y respectively, such that $f(\mathcal{U}) \subset \mathcal{V}$. By 2.6.1, $\widehat{\Omega}^m_f\big|_{\mathcal{U}} \cong (\widehat{\Omega}^m_{S/R})^{\sim_S}$, whence $\widehat{\Omega}^m_{f,x} \cong \widehat{\Omega}^m_{S/R} \otimes_S B$. There are then natural isomorphisms

$$\widehat{\Omega}^m_{f,x} \otimes_B \widehat{B} \xrightarrow{\ \sim\ } (\widehat{\Omega}^m_{S/R} \otimes_S B) \otimes_B \widehat{B} \xrightarrow{\ \sim\ } \widehat{\Omega}^m_{S/R} \otimes_S \widehat{B} \xrightarrow{\ \alpha\ } \widehat{\Omega}^m_{\widehat{B}/R} \xrightarrow{\ \beta\ } \widehat{\Omega}^m_{\widehat{B}/\widehat{A}},$$

where α is the isomorphism which follows from 2.5.2, applied to the sequence $R \to S \to \widehat{B}$ (by the remark preceding 2.6.9, $S \to \widehat{B}$ is smooth), while β is the isomorphism obtained similarly from the sequence $R \to \widehat{A} \to \widehat{B}$. □

Corollary 2.6.10. *With notation and assumptions as in* 2.6.9, *assume further that $f\colon (\mathcal{X}, \Delta_1) \to (\mathcal{Y}, \Delta)$ is a smooth map in $\mathbb{F}_c$ (see section* 2.1*). Let d be the relative dimension of f at x, $p := \Delta_1(x)$, and $q := \Delta(y)$. Let m_A be the maximal ideal of A. Then*

$$\dim(B/m_A B) = p + d - q.$$

Proof. By 2.4.5(ii), B is a formally smooth A-algebra; and so by(v) and (vi) of 2.4.1, $\widehat{B}$ is a formally smooth $\widehat{A}$-algebra. By 2.6.9, $\widehat{\Omega}^1_{\widehat{B}/\widehat{A}}$ is free of rank d, and by definition of $\mathbb{F}_c$,

$$q - p = \Delta(y) - \Delta_1(x) = \mathrm{tr.deg.}_{k(y)} k(x).$$

So by ([**13**, Lemma 3.9]),

$$\dim(B/m_A B) = \dim(\widehat{B}/m_A \widehat{B}) = d - (q - p).$$

□

3. Local cohomology and Cousin complexes

In §3.1 we recall some basic facts about local cohomology modules and about various relations between different definitions of local cohomology in the presence of quasi-coherence and torsionness conditions. In §3.2 we review basic definitions and properties of Cousin complexes, and of the Cousin functor E from the derived category to Cousin complexes. (For smooth formal-scheme maps $\mathcal{X} \to \mathcal{Y}$, E will play an important role in the construction of a functor from Cousin complexes over $\mathcal{Y}$ to Cousin complexes over $\mathcal{X}$.) The final subsection is on Cohen-Macaulay complexes, the derived category counterparts of Cousin complexes. The results there are based on Suominen's work [**25**], in which it is shown that E induces an equivalence of categories from the category of Cohen-Macaulay complexes to the category of Cousin complexes (where the latter, and hence the former, is abelian).

3.1. Local cohomology. Let X be a topological space, Z a closed subset, and $\mathcal{F}$ an abelian sheaf on X. We denote by $\varGamma_Z\mathcal{F}$ the subsheaf of $\mathcal{F}$ whose sections over any open $U \subset X$ are those elements of $\mathcal{F}(U)$ whose support lies in Z. Recall further, from §2.3, the definition of the functor $\varGamma_{\mathcal{I}}$ for a ringed space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-ideal $\mathcal{I}$, and of the functor $\varGamma_{\mathfrak{a}}$ for an ideal $\mathfrak{a}$ in a ring A.

Lemma 3.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $x \in X$ a point. If the $\mathcal{O}_X$-ideal $\mathcal{I}$ is locally finitely-generated, then $(\varGamma_{\mathcal{I}}\mathcal{F})_x = \varGamma_{\mathcal{I}_x}\mathcal{F}_x$, where each side of the equality has been identified naturally with an $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$.*

Proof. The equality results from the natural isomorphisms

$$\begin{aligned}\big(\varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{F})\big)_x &\xrightarrow{\ \sim\ } \varinjlim_n \big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{F})\big)_x \\ &\xrightarrow{\ \sim\ } \varinjlim_n \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathcal{I}_x^n, \mathcal{F}_x),\end{aligned}$$

the second isomorphism holding because locally, $\mathcal{O}_X/\mathcal{I}^n$ has finite presentation. $\square$

It follows that if Z is the support of $\mathcal{O}_X/\mathcal{I}$ then for any $\mathcal{F} \in \mathcal{A}(X)$ there is a natural inclusion $\varGamma_{\mathcal{I}}\mathcal{F} \subset \varGamma_Z\mathcal{F}$. In general, this inclusion is not an equality.

Lemma 3.1.2. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a noetherian formal scheme and $Z \subset \mathcal{X}$ a closed subset. Let $\mathcal{I}$ be an open coherent $\mathcal{O}_{\mathcal{X}}$-ideal such that the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is Z. Then for any $\mathcal{F} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ we have*

$$\varGamma_{\mathcal{I}}\mathcal{F} = \varGamma_Z\mathcal{F}.$$

Proof. It suffices to show that for any affine open subset $\mathcal{U} := \operatorname{Spf}(A)$ we have $(\varGamma_Z\mathcal{F})(\mathcal{U}) \subset (\varGamma_{\mathcal{I}}\mathcal{F})(\mathcal{U})$, the opposite containment having been noted above. Set $F = \mathcal{F}(\mathcal{U})$ and $I = \mathcal{I}(\mathcal{U})$. By 2.3.4(iii), $\mathcal{F} \in \mathcal{A}_{\vec{c}}(\mathcal{X})$ and $\mathcal{F}|_{\mathcal{U}} \cong F^{\sim}$, whence by 2.3.3 and 2.2.1(iv), $(\varGamma_{\mathcal{I}}\mathcal{F})(\mathcal{U}) = \varGamma_I F$. In particular, since $\mathcal{F}$ is a torsion $\mathcal{O}_{\mathcal{X}}$-module, this equality for an ideal of definition shows that F is a torsion A-module. So by 2.2.1(iv) and 2.3.1, for any $s \in A$, with $\mathcal{U}_s := \operatorname{Spf}(A_{\{s\}})$ we have $\mathcal{F}(\mathcal{U}_s) = F_s$. If $s \in I$, then $\mathcal{U}_s \cap Z = \emptyset$, so the image in F_s of any $f \in (\varGamma_Z\mathcal{F})(\mathcal{U})$ is zero, i.e., any such f is annihilated by a power of s, and hence, I being finitely generated, by a power of I. Thus $(\varGamma_Z\mathcal{F})(\mathcal{U}) \subset \varGamma_I F = (\varGamma_{\mathcal{I}}\mathcal{F})(\mathcal{U})$, as desired. $\square$

For an abelian sheaf $\mathcal{F}$ on a topological space X, and $x \in X$, let $\varGamma_x\mathcal{F} \subset \mathcal{F}_x$ be the stalk at x of the sheaf $\varGamma_Z\mathcal{F}$ with $Z := \overline{\{x\}}$, the closure in $\mathcal{X}$ of the set $\{x\}$.

Corollary 3.1.3. *Let $\mathfrak{X}$ be a noetherian formal scheme, and $\mathcal{F} \in \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$. For any $x \in \mathfrak{X}$ let m_x be the maximal ideal of the local ring $\mathcal{O}_{\mathfrak{X},x}$. Then*

$$\varGamma_{m_x}\mathcal{F}_x = \varGamma_x\mathcal{F} \subset \mathcal{F}_x.$$

Proof. Let $\mathfrak{I}$ be the largest coherent ideal defining $Z = \overline{\{x\}}$. Then $\mathfrak{I}$ is an open ideal and the stalk of $\mathfrak{I}$ at x is m_x (since this holds true over the scheme obtained by reducing modulo an ideal of definition of $\mathfrak{X}$). By 3.1.1 and 3.1.2 we have $\varGamma_{m_x}\mathcal{F}_x = (\varGamma_{\mathfrak{I}}\,\mathcal{F})_x = (\varGamma_Z\mathcal{F})_x$. □

The functors $\varGamma_?$ defined above are all left exact. We denote by $\mathbf{R}\varGamma_?$ the corresponding right-derived functors and by $H^i_? := H^i\mathbf{R}\varGamma_?$ the corresponding i-th right-derived functors.

Z For a complex $\mathcal{F}^\bullet$ on a ringed space $(X, \mathcal{O}_X)$, $H^i_?\mathcal{F}^\bullet := H^i\mathbf{R}\varGamma_?\,\mathcal{F}^\bullet$ is the *abelian group* sometimes referred to in the literature as "local hypercohomology with supports in ?" and sometimes denoted "$\mathbb{H}^i\dots$" The symbol "$H^i_?\mathcal{F}^\bullet$" might also, conceivably, denote the *complex* whose n-th term is $H^i_?\mathcal{F}^n$, differentials being the natural induced ones; in this paper, that will never be so.

We now review the construction of local cohomology via "direct-limit Koszul complexes". Let $(X, \mathcal{O}_X)$ be a ringed space. For an element $t \in \Gamma(X, \mathcal{O}_X)$, let $\mathcal{K}^\bullet(t)$ be the complex which in degrees 0 and 1 is multiplication by t from $\mathcal{K}^0(t) := \mathcal{O}_X$ to $\mathcal{K}^1(t) := \mathcal{O}_X$, and is zero elsewhere. For $0 \le r \le s$, there is a map of complexes $\mathcal{K}^\bullet(t^r) \to \mathcal{K}^\bullet(t^s)$ which is identity in degree 0 and multiplication by t^{s-r} in degree 1. Thus we get a direct system of complexes, whose direct limit we denote by $\mathcal{K}^\bullet_\infty(t)$. For any sequence $\mathbf{t} = (t_1, ..., t_n)$ of elements in $\Gamma(X, \mathcal{O}_X)$ we set (with $\otimes = \otimes_{\mathcal{O}_X}$)

$$\mathcal{K}^\bullet_\infty(\mathbf{t}) := \mathcal{K}^\bullet_\infty(t_1) \otimes \dots \otimes \mathcal{K}^\bullet_\infty(t_n);$$

and for any complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$ modules set $\mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}^\bullet) := \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{F}^\bullet$. With $\mathfrak{I}$ the $\mathcal{O}_X$-ideal generated by the sequence $\mathbf{t}$, there are natural identifications

$$\varGamma_{\mathfrak{I}}\,\mathcal{F}^j = \ker\big(\mathcal{K}^0_\infty(\mathbf{t}, \mathcal{F}^j) \to \mathcal{K}^1_\infty(\mathbf{t}, \mathcal{F}^j)\big) \qquad (j \in \mathbb{Z}),$$

yielding a map of complexes $\varGamma_{\mathfrak{I}}\,\mathcal{F}^\bullet \to \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}^\bullet)$, whose composition with the natural map $\mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}^\bullet) = \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{F}^\bullet \to \mathcal{K}^0_\infty(\mathbf{t}) \otimes \mathcal{F}^\bullet = \mathcal{F}^\bullet$ is the inclusion $\varGamma_{\mathfrak{I}}\,\mathcal{F}^\bullet \hookrightarrow \mathcal{F}^\bullet$. Note that the stalk of $\mathcal{K}^\bullet_\infty(t)$ at any point $x \in X$ looks like the localization map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}[1/t]$. It follows that $\mathcal{K}^\bullet_\infty(\mathbf{t})$ is a complex of flat $\mathcal{O}_X$-modules. In particular, $\mathcal{K}^\bullet_\infty(\mathbf{t}, -)$ takes quasi-isomorphisms to quasi-isomorphisms and hence induces a functor $\mathbf{D}(X) \to \mathbf{D}(X)$, also denoted by $\mathcal{K}^\bullet_\infty(\mathbf{t}, -)$. In fact $\mathcal{K}^\bullet_\infty(\mathbf{t}, -)$ is a δ-functor (see §1.4, (vii)), with $\Theta\colon \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}^\bullet[1]) \xrightarrow{\sim} (\mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{F}^\bullet))[1]$ given in degree $p+q$ by $(-1)^p$ times the identity map of $\mathcal{K}^p_\infty(\mathbf{t}) \otimes \mathcal{F}^{q+1}$.

Proposition 3.1.4. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a noetherian formal scheme, and $\mathbf{t}$ a finite sequence in $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Let $\mathfrak{I}$ be the $\mathcal{O}_{\mathfrak{X}}$-ideal generated by $\mathbf{t}$. Then for any complex of $\mathcal{A}(\mathfrak{X})$-injectives $\mathcal{L}^\bullet$, the natural map $\varGamma_{\mathfrak{I}}\,\mathcal{L}^\bullet \to \mathcal{K}^\bullet_\infty(\mathbf{t}, \mathcal{L}^\bullet)$ is a quasi-isomorphism. Moreover, the induced natural isomorphism of functors*

$$\mathbf{R}\varGamma_{\mathfrak{I}} \xrightarrow{\sim} \mathcal{K}^\bullet_\infty(\mathbf{t}, -)$$

is an isomorphism of δ-functors. Hence there is a δ-functorial isomorphism

$$\mathbf{R}\varGamma_{\mathfrak{I}}\,\mathcal{F}^\bullet \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathfrak{I}}\,\mathcal{O}_{\mathfrak{X}} \underset{=}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}^\bullet \qquad (\mathcal{F}^\bullet \in \mathbf{D}(\mathfrak{X}))$$

whose composition with the natural map $\mathbf{R}\varGamma_{\mathfrak{I}}\,\mathcal{O}_{\mathfrak{X}} \underset{=}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}^\bullet \to \mathcal{O}_{\mathfrak{X}} \underset{=}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}^\bullet \cong \mathcal{F}^\bullet$ is the natural map $\mathbf{R}\varGamma_{\mathfrak{I}}\,\mathcal{F}^\bullet \to \mathcal{F}^\bullet$.

PROOF. See [**1**, Lemma 3.1.1, (1) ⇒ (2)]. Verification of the δ-part is straightforward. (The natural δ-structure on $\mathbf{R}\varGamma_{\mathcal{J}}$ can be extracted from [**17**, Example 2.2.4].) The rest is left to the reader. □

Before stating some consequences of 3.1.4 we note that the constructions above have obvious analogs over rings. For a ring A and a sequence $\mathbf{t}$ in A, the direct-limit Koszul complex $K^\bullet_\infty(\mathbf{t})$ of A-modules is defined in a manner similar to the above. If A is noetherian and $I^\bullet$ a complex of A-injectives, then the natural map $\varGamma_{\mathbf{t}A} I^\bullet \to K^\bullet_\infty(\mathbf{t}, I^\bullet)$ is a quasi-isomorphism.

LEMMA 3.1.5. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a noetherian formal scheme.*

(i) *For any coherent $\mathcal{O}_{\mathcal{X}}$-ideals $\mathcal{I}, \mathcal{J}$, the natural map $\mathbf{R}\varGamma_{\mathcal{I}+\mathcal{J}} \to \mathbf{R}\varGamma_{\mathcal{I}}\,\mathbf{R}\varGamma_{\mathcal{J}}$ is an isomorphism.*

(ii) *For any sequences $\mathbf{t}, \mathbf{s}$ in $G := \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $\mathbf{s}G \subset \sqrt{\mathbf{t}G}$, and for any complex $\mathcal{F}^\bullet \in \mathbf{C}(\mathcal{X})$, the map (with $\otimes = \otimes_{\mathcal{O}_{\mathcal{X}}}$)*

$$\mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{K}^\bullet_\infty(\mathbf{s}) \otimes \mathcal{F}^\bullet \to \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{F}^\bullet$$

induced by the natural map $\mathcal{K}^\bullet_\infty(\mathbf{s}) \to \mathcal{O}_{\mathcal{X}}$ is a quasi-isomorphism.

(iii) *For any $x \in \mathcal{X}$ and $\mathcal{F}^\bullet \in \mathbf{D}(\mathcal{X})$ the natural map $(\mathbf{R}\varGamma_{\mathcal{I}}\,\mathcal{F}^\bullet)_x \to \mathbf{R}\varGamma_{\mathcal{I}_x}\mathcal{F}^\bullet_x$ is a $\mathbf{D}(\mathcal{O}_{\mathcal{X},x})$-isomorphism.*

PROOF. (i). One argument is given in the last four lines on page 25 of [**1**]. Another is that the map in question is the canonical isomorphism resulting from the fact that both source and target are right adjoints of the inclusion into $\mathbf{D}(\mathcal{X})$ of the full subcategory whose objects are complexes whose homology is $(\mathcal{I}+\mathcal{J})$-torsion, see [**2**, p. 49, Prop. 5.2.1(c)]. One can also use 3.1.4, as follows.

The assertion being local, we may assume that $\mathcal{X}$ is affine, so that there are sequences $\mathbf{t}, \mathbf{s}$ in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ generating $\mathcal{I}, \mathcal{J}$ respectively. For any $\mathcal{F}^\bullet \in \mathbf{D}(\mathcal{X})$, we claim that the following natural diagram commutes:

$$\begin{array}{ccc}
\mathbf{R}\varGamma_{\mathcal{I}+\mathcal{J}}\mathcal{F}^\bullet & \longrightarrow & \mathbf{R}\varGamma_{\mathcal{I}}\,\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{F}^\bullet \\
\big\downarrow & & \big\downarrow \\
\mathcal{K}^\bullet_\infty(\mathbf{t},\mathbf{s}) \otimes \mathcal{F}^\bullet & \longrightarrow & \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{K}^\bullet_\infty(\mathbf{s}) \otimes \mathcal{F}^\bullet
\end{array}$$

The bottom map is an isomorphism, as are the vertical maps (by 3.1.4), and the assertion results.

Verification of commutativity is straightforward once the diagram is expanded as in (3.1.5.1) below—where $\mathcal{F}^\bullet$ is assumed, w.l.o.g., to be K-injective, $\mathcal{K}_{\mathbf{t}}$ is an abbreviation for $\mathcal{K}^\bullet_\infty(\mathbf{t})$, etc., and $\mathcal{K}_{\mathbf{s}} \otimes \mathcal{F}^\bullet \to \mathcal{E}^\bullet$ is a K-injective resolution (see [**1**, p. 19, proof of $(2)' \Rightarrow (2)$]).

(ii). Let $\mathcal{I}, \mathcal{J}$ be the $\mathcal{O}_{\mathcal{X}}$-ideals generated by $(\mathbf{t}), (\mathbf{t}, \mathbf{s})$ respectively. Note that $\varGamma_{\mathcal{I}} = \varGamma_{\mathcal{J}}$. Let $\mathcal{L}^\bullet$ be a K-injective resolution of $\mathcal{F}^\bullet$. In the commutative diagram

$$\begin{array}{ccccc}
\mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{K}^\bullet_\infty(\mathbf{s}) \otimes \mathcal{F}^\bullet & \longrightarrow & \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{K}^\bullet_\infty(\mathbf{s}) \otimes \mathcal{L}^\bullet & \longleftarrow & \varGamma_{\mathcal{J}}\,\mathcal{L}^\bullet \\
\big\downarrow & & \big\downarrow & & \big\| \\
\mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{F}^\bullet & \longrightarrow & \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes \mathcal{L}^\bullet & \longleftarrow & \varGamma_{\mathcal{I}}\,\mathcal{L}^\bullet
\end{array}$$

the horizontal maps are quasi-isomorphisms, and the result follows.

(3.1.5.1)

$$
\begin{tikzcd}
\varGamma_{\mathcal{I}+\mathcal{J}}\mathcal{F}^\bullet \arrow[r, equal] \arrow[d, equal] & \varGamma_{\mathcal{I}}\,\varGamma_{\mathcal{J}}\,\mathcal{F}^\bullet \arrow[r] \arrow[ddd, equal] & \mathbf{R}\varGamma_{\mathcal{I}}\,\varGamma_{\mathcal{J}}\,\mathcal{F}^\bullet \arrow[d, equal] & \\
\mathbf{R}\varGamma_{\mathcal{I}+\mathcal{J}}\mathcal{F}^\bullet \arrow[ddd] & & \mathbf{R}\varGamma_{\mathcal{I}}\,\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{F}^\bullet \arrow[d] & \\
& & \mathbf{R}\varGamma_{\mathcal{I}}\,(\mathcal{K}_{\mathbf{s}}\otimes\mathcal{F}^\bullet) \arrow[r] & \mathbf{R}\varGamma_{\mathcal{I}}\,\mathcal{E}^\bullet \arrow[d, equal] \\
& \varGamma_{\mathcal{I}}\,\varGamma_{\mathcal{J}}\,\mathcal{F}^\bullet \arrow[r] \arrow[d] & \varGamma_{\mathcal{I}}\,(\mathcal{K}_{\mathbf{s}}\otimes\mathcal{F}^\bullet) \arrow[r] \arrow[u] \arrow[d] & \varGamma_{\mathcal{I}}\,\mathcal{E}^\bullet \arrow[d] \\
\mathcal{K}_{\mathbf{t},\mathbf{s}}\otimes\mathcal{F}^\bullet \arrow[r, equal] & \mathcal{K}_{\mathbf{t},\mathbf{s}}\otimes\mathcal{F}^\bullet \arrow[r, equal] & \mathcal{K}_{\mathbf{t}}\otimes\mathcal{K}_{\mathbf{s}}\otimes\mathcal{F}^\bullet & \mathcal{K}_{\mathbf{t}}\otimes\mathcal{E}^\bullet \arrow[l, "\sim"']
\end{tikzcd}
$$

(iii). The assertion being local, we may assume that $\mathcal{X}$ is affine, so that there is a sequence $\mathbf{t}$ in $\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ generating $\mathcal{I}$. We may assume further that $\mathcal{F}^\bullet$ is K-injective. The following natural $\mathbf{D}(\mathcal{O}_{\mathcal{X},x})$-diagram—where $K^\bullet_\infty(\mathbf{t})$ is the direct-limit Koszul complex on $\mathbf{t}$ over $\mathcal{O}_{\mathcal{X},x}$, $\mathcal{K}_{\mathbf{t}}$ is an abbreviation for $\mathcal{K}^\bullet_\infty(\mathbf{t})$, etc., and $\mathcal{F}^\bullet_x \to E^\bullet$ is a K-injective $\mathcal{O}_{\mathcal{X},x}$-resolution—commutes:

$$
\begin{tikzcd}
(\mathbf{R}\varGamma_{\mathcal{I}}\,\mathcal{F}^\bullet)_x = (\varGamma_{\mathcal{I}}\,\mathcal{F}^\bullet)_x \arrow[r, "\sim", "3.1.1"'] \arrow[d, "\simeq"'] & \varGamma_{\mathcal{I}_x}\mathcal{F}^\bullet_x \arrow[r] \arrow[d] & \varGamma_{\mathcal{I}_x}E^\bullet = \mathbf{R}\varGamma_{\mathcal{I}_x}\mathcal{F}^\bullet_x \arrow[d, "\simeq"] \\
(\mathcal{K}_{\mathbf{t}}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{F}^\bullet)_x \arrow[r, "\sim"] & K_{\mathbf{t}}\otimes_{\mathcal{O}_{\mathcal{X},x}}\mathcal{F}^\bullet_x & K_{\mathbf{t}}\otimes_{\mathcal{O}_{\mathcal{X},x}}E^\bullet \arrow[l, "\sim"']
\end{tikzcd}
$$

The maps in the bottom row are isomorphisms, as are the outside vertical maps (by 3.1.4 and its analog over rings), and the assertion results. $\square$

LEMMA 3.1.6. *Let $f\colon \mathcal{X}\to\mathcal{Y}$ be a map of noetherian formal schemes, $\mathcal{G}^\bullet\in\mathbf{D}(\mathcal{Y})$ and $\mathcal{L}^\bullet\in\mathbf{D}(\mathcal{X})$. Let $\mathcal{I},\mathcal{J}$ be coherent ideals in $\mathcal{O}_{\mathcal{X}},\mathcal{O}_{\mathcal{Y}}$ respectively with $\mathcal{J}\mathcal{O}_{\mathcal{X}}\subset\mathcal{I}$. Then the natural map is an isomorphism*

$$\mathbf{R}\varGamma_{\mathcal{I}}\,(\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{G}^\bullet \underline{\underline{\otimes}}_{\mathcal{Y}}\,\mathcal{L}^\bullet) \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathcal{I}}\,(\mathbf{L}f^*\mathcal{G}^\bullet \underline{\underline{\otimes}}_{\mathcal{X}}\,\mathcal{L}^\bullet).$$

PROOF. Using 3.1.4 and commutativity of $\mathbf{L}f^*$ and $\underline{\underline{\otimes}}$ we reduce to showing that the natural map is an isomorphism $\mathbf{R}\varGamma_{\mathcal{I}}\,(\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{O}_{\mathcal{Y}}) \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathcal{I}}\,(\mathcal{O}_{\mathcal{X}})$. By [**2**, p. 53, Prop. 5.2.8(b)], the map $\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{O}_{\mathcal{Y}}\to\mathcal{O}_{\mathcal{X}}$ factors naturally as

$$\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{O}_{\mathcal{Y}} \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathcal{J}\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$$

(the first map an isomorphism). The conclusion follows then from 3.1.5(i). $\square$

LEMMA 3.1.7. *In the situation of* 3.1.4, *let $\mathcal{I}^\bullet$ be a complex of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injectives. Then the natural map $\varGamma_{\mathcal{I}}\,\mathcal{I}^\bullet\to\mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{I}^\bullet)$ is a quasi-isomorphism. In particular, for any quasi-isomorphism $\mathcal{I}^\bullet\to\mathcal{L}^\bullet$ with $\mathcal{L}^\bullet$ a complex of $\mathcal{A}(\mathcal{X})$-injectives, the natural map $\varGamma_{\mathcal{I}}\,\mathcal{I}^\bullet\to\varGamma_{\mathcal{I}}\,\mathcal{L}^\bullet$ is a quasi-isomorphism.*

Proof. (Sketch) Since it suffices to check that over affine open subsets of $\mathcal{X}$ the natural map $\Psi\colon \varGamma_{\mathcal{J}}\mathcal{I}^\bullet \to \mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{I}^\bullet)$ is a quasi-isomorphism, we may assume that $\mathcal{X}=\mathcal{U}=\mathrm{Spf}(A)$ for some noetherian adic ring A. Note that Ψ is obtained by applying the functor $\sim$ of 2.2.1 to the natural map $\psi\colon \varGamma_{\mathbf{t}A}I^\bullet \to K^\bullet_\infty(\mathbf{t},I^\bullet)$ where $I^\bullet=\Gamma(\mathcal{U},\mathcal{I}^\bullet)$. Since $\mathcal{I}^\bullet$ consists of $\mathcal{A}_{\mathrm{qct}}(\mathcal{U})$-injectives, and so by 2.3.6(ii) is a direct sum of sheaves of the form $i_xJ_x=J_x^\sim$ (see proof of 2.3.5, (i)$\Longrightarrow$(iii)), therefore $I^\bullet$ is a direct sum of A-modules of the form J_x, hence consists of A-injectives. Thus, by the ring-theoretic version of 3.1.4, ψ is a quasi-isomorphism, whence so is Ψ. $\square$

An alternate way of proving the above lemma is as follows. By 2.3.6(ii), any *indecomposable* injective $\mathcal{I}$ in $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ is an $\mathcal{A}(\mathcal{X})$-injective, and so by 3.1.4 the natural map $\Psi\colon \varGamma_{\mathcal{J}}\mathcal{I} \to \mathcal{K}^\bullet_\infty(\mathbf{t},\mathcal{I})$ is a quasi-isomorphism. Since Ψ behaves well with respect to direct sums ($\mathcal{O}_{\mathcal{X}}/\mathcal{J}^n$ being coherent, the functor $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}/\mathcal{J}^n,-)$ commutes with direct sums) therefore we can extend the result to any $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injective $\mathcal{I}$ (see again 2.3.6(ii)). Finally using way-out type arguments we extend the result to any complex $\mathcal{I}^\bullet$ of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injectives (cf. [**1**, p. 22]).

Proposition 3.1.8. *Let $\mathcal{X}$ be a noetherian formal scheme.*

(i) *For any $\mathcal{F}^\bullet\in\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$, we have $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet\in\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$. For any $\mathcal{F}^\bullet\in\mathbf{D}_{\mathrm{t}}(\mathcal{X})$, the canonical map $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet\to\mathcal{F}^\bullet$ is an isomorphism.*

(ii) *The natural functors $\mathbf{D}^+(\mathcal{A}_{\mathrm{qct}}(\mathcal{X}))\to\mathbf{D}^+_{\mathrm{qc}}(\mathcal{A}_{\mathrm{t}}(\mathcal{X}))\to\mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$ are equivalences of categories. In particular, any $\mathcal{F}^\bullet\in\mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$ is isomorphic to a bounded below complex of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injectives.*

Proof. For (i) see [**1**, p. 49, 5.2.1]. For (ii), see [**27**, Theorem 4.8] (or [**2**, p. 57, 5.3.1] for an unbounded version). $\square$

Proposition 3.1.9. *Let $\mathcal{X}$ be a noetherian formal scheme. Let Z be a closed subset of $\mathcal{X}$ and $\mathcal{J}$ an open coherent $\mathcal{O}_{\mathcal{X}}$-ideal such that $Z=\mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{J})$.*

(i) *For $\mathcal{F}^\bullet\in\mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$ the natural map is an isomorphism $\mathbf{R}\varGamma_{\mathcal{J}}\mathcal{F}^\bullet\xrightarrow{\sim}\mathbf{R}\varGamma_Z\mathcal{F}^\bullet$.*

(ii) *For any $\mathcal{F}^\bullet\in\mathbf{D}^+_{\mathrm{qc}}(\mathcal{X})$, the natural maps are isomorphisms*

$$\mathbf{R}\varGamma_{\mathcal{J}}\mathcal{F}^\bullet \xleftarrow{\sim} \mathbf{R}\varGamma_{\mathcal{J}}\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet \xrightarrow{\sim} \mathbf{R}\varGamma_Z\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet.$$

Proof. (i). By 3.1.8(ii), we may assume that $\mathcal{F}^\bullet$ is a bounded-below complex of $\mathcal{A}_{\mathrm{qct}}$-injectives. Let $\mathcal{F}^\bullet\to\mathcal{L}^\bullet$ be an $\mathcal{A}(\mathcal{X})$-injective resolution of $\mathcal{F}^\bullet$. In the commutative $\mathbf{D}(\mathcal{X})$-diagram

$$\begin{array}{ccc}
\varGamma_{\mathcal{J}}\mathcal{F}^\bullet & \xrightarrow{a} & \varGamma_{\mathcal{J}}\mathcal{L}^\bullet\cong\mathbf{R}\varGamma_{\mathcal{J}}\mathcal{F}^\bullet \\
{\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle d} \\
\varGamma_Z\mathcal{F}^\bullet & \xrightarrow[c]{} & \varGamma_Z\mathcal{L}^\bullet\cong\mathbf{R}\varGamma_Z\mathcal{F}^\bullet
\end{array}$$

the map a is an isomorphism by 3.1.7, b is an isomorphism by 3.1.2, and c is an isomorphism because $\mathcal{F}^\bullet$ consists of flasque sheaves (see 2.3.6(iii)), which are $\varGamma_Z$-acyclic. Hence d is an isomorphism.

(ii). Since $\mathcal{J}$ is open, the first isomorphism is given by 3.1.5(i). By 3.1.8(i), $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet\in\mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$, and so the second isomorphism is given by (i). $\square$

Let $\mathcal{X}$ be a noetherian formal scheme and $\mathcal{F}^\bullet \in \mathbf{D}^+_{\mathrm{qc}}(\mathcal{X})$. For any $x \in \mathcal{X}$, set $Z := \overline{\{x\}}$ and let $\mathcal{I}$ be the largest coherent ideal defining Z, so that $\mathcal{I}$ is open and the stalk $\mathcal{I}_x$ of $\mathcal{I}$ at x is the maximal ideal m_x of $\mathcal{O}_{\mathcal{X},x}$. Then 3.1.9(ii) and 3.1.5(iii) give natural isomorphisms

$$\mathbf{R}\varGamma_x \mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet = (\mathbf{R}\varGamma_Z \mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F}^\bullet)_x \cong (\mathbf{R}\varGamma_{\mathcal{I}}\, \mathcal{F}^\bullet)_x \cong \mathbf{R}\varGamma_{m_x} \mathcal{F}^\bullet_x. \tag{6}$$

LEMMA 3.1.10. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of noetherian formal schemes. Then*:

(i) $f^*(\mathcal{A}_{\mathrm{qc}}(\mathcal{Y})) \subset \mathcal{A}_{\mathrm{qc}}(\mathcal{X})$.

(ii) *The category $\mathcal{A}_{\mathrm{qc}}(\mathcal{X})$ is closed under tensor products.*

(iii) *For any sequence $\mathbf{t}$ in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and $j \geq 0$, we have $\mathcal{K}^j_\infty(\mathbf{t}) \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})$.*

PROOF. (i) holds for any map of ringed spaces.
(ii) holds over any ringed space.
(iii) holds because $\mathcal{K}^j_\infty(\mathbf{t})$ is a $\varinjlim$ of finitely-generated free $\mathcal{O}_{\mathcal{X}}$-modules. $\square$

LEMMA 3.1.11. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of noetherian formal schemes, $y \in \mathcal{Y}$, and M a zero-dimensional $\mathcal{O}_{\mathcal{Y},y}$-module. Let $\mathcal{M} = i_y M$ be the corresponding $\mathcal{O}_{\mathcal{Y}}$-module* (2.3.5). *Suppose $\mathcal{G} \in \mathcal{A}_{\mathrm{qc}}(\mathcal{X})$. Then for any $x \in \mathcal{X}$ such that $f(x) \neq y$, and any $j \geq 0$,*

$$H^j_x \mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M} \otimes_{\mathcal{X}} \mathcal{G}) = 0 = H^j_{m_x}(f^*\mathcal{M} \otimes_{\mathcal{X}} \mathcal{G})_x.$$

PROOF. Set $\mathcal{F} := f^*\mathcal{M} \otimes_{\mathcal{X}} \mathcal{G}$. By 2.3.5, $\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$, and hence by parts (i) and (ii) of 3.1.10, $\mathcal{F} \in \mathcal{A}_{\mathrm{qc}}(\mathcal{X})$. By (6), there is an isomorphism $H^j_x \mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{F} \cong H^j_{m_x}\mathcal{F}_x$ with $m_x$the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$. We consider two cases:

(a) When $f(x) \notin \overline{\{y\}}$, the stalk of $\mathcal{M}$ at $f(x)$ is 0, so $(f^*\mathcal{M})_x = 0$, $\mathcal{F}_x = 0$, and $H^j_{m_x}\mathcal{F}_x = 0$.

(b) When $f(x) \in \overline{\{y\}}$, there exists a non-unit $t \in \mathcal{O}_{\mathcal{Y},f(x)}$ whose image under the natural map $\mathcal{O}_{\mathcal{Y},f(x)} \to \mathcal{O}_{\mathcal{Y},y}$ is a unit in $\mathcal{O}_{\mathcal{Y},y}$. Then t acts invertibly on M (since M is an $\mathcal{O}_{\mathcal{Y},y}$-module) and hence on $\mathcal{F}_x$ and hence on $H^j_{m_x}\mathcal{F}_x$. But the image of t under the natural local homomorphism $\mathcal{O}_{\mathcal{Y},f(x)} \to \mathcal{O}_{\mathcal{X},x}$ lies in m_x and therefore every element of the m_x-torsion module $H^j_{m_x}\mathcal{F}_x$ is annihilated by a power of t. Thus $H^j_{m_x}\mathcal{F}_x = 0$. $\square$

3.2. Cousin complexes. We will use the notion of Cousin complex as in [**11**, Chap. IV]. (Additional properties in a more general context may be found in [**25**].) We first review the relevant definitions.

Throughout this subsection X denotes a noetherian topological space in which every irreducible closed subset has a unique generic point; and X is assumed to be equipped with a filtration

$$Z^\bullet:\quad \cdots \supseteq Z^{p-1} \supseteq Z^p \supseteq Z^{p+1} \cdots \qquad (Z^p \subset X) \tag{7}$$

satisfying the following conditions (cf. [**11**, p. 240]):

(a) It is strictly exhaustive, i.e., $Z^p = X$ for some $p \in \mathbb{Z}$.

(b) It is separated, i.e., $\bigcap_p Z^p = \emptyset$.

(c) Each Z^p is stable under specialization.

(d) For any p, if $x \rightsquigarrow x'$ is a specialization and $x, x' \in Z^p \setminus Z^{p+1}$ then $x = x'$.

Corresponding to the filtration $Z^\bullet$ there is a filtration by subfunctors of the identity functor:

$$\Gamma^\bullet:\quad \ldots \supseteq \varGamma_{Z^{p-1}} \supseteq \varGamma_{Z^p} \supseteq \varGamma_{Z^{p+1}} \ldots$$

Recall that for $Z \subset X$ and $\mathcal{F}$ an abelian sheaf on X, $\varGamma_Z\mathcal{F}$ is the sheaf of sections of $\mathcal{F}$ with support in Z, so that $\varGamma_Z$ is an idempotent left exact functor. Recall also that for $x \in X$, $\varGamma_x\mathcal{F}$ is the stalk at x of $\varGamma_{\overline{\{x\}}}\mathcal{F}$. One sets $\varGamma_{Z^p/Z^{p+1}}\mathcal{F} := \varGamma_{Z^p}\mathcal{F}/\varGamma_{Z^{p+1}}\mathcal{F}$.

In [**11**, p. 226] there is defined, for any flasque abelian sheaf $\mathcal{F}$ on X, an isomorphism (with i_x as in 2.3.5)

$$\varGamma_{Z^p/Z^{p+1}}\mathcal{F} \;\xrightarrow{\;\sim\;}\; \bigoplus_{x\in Z^p\setminus Z^{p+1}} i_x(\varGamma_x\mathcal{F}). \tag{8}$$

This isomorphism arises thus: for any open $U \subset X$ and $\xi \in (\varGamma_{Z^p}\mathcal{F})(U)$, the support of ξ, a closed subset of $Z^p \cap U$, contains only finitely many $x \in Z^p \setminus Z^{p+1}$; hence the stalk of $\varGamma_{Z^p}\mathcal{F}$ at any such x is $\varGamma_x\mathcal{F}$, and the resulting natural map

$$\varGamma_{Z^p}\mathcal{F} \longrightarrow \prod_{x\in Z^p\setminus Z^{p+1}} i_x(\varGamma_x\mathcal{F})$$

has kernel $\varGamma_{Z^{p+1}}\mathcal{F}$ and image $\oplus_{\{x\in Z^p\setminus Z^{p+1}\}}\, i_x(\varGamma_x\mathcal{F})$.

As homology commutes with direct sums (X being noetherian), and i_x is an exact functor, we can replace any homologically bounded-below abelian complex $\mathcal{F}^\bullet$ on X by a flasque resolution[4] and deduce a canonical functorial isomorphism

$$H^n_{Z^p/Z^{p+1}}\mathcal{F}^\bullet := H^n\mathbf{R}\varGamma_{Z^p/Z^{p+1}}\mathcal{F}^\bullet \;\xrightarrow{\;\sim\;}\; \bigoplus_{x\in Z^p\setminus Z^{p+1}} i_x(H^n_x\mathcal{F}^\bullet). \tag{8$'$}$$

An abelian sheaf $\mathcal{G}$ is said to *lie on the* Z^p/Z^{p+1}*–skeleton of* X if either of the following two equivalent conditions are satisfied ([**11**, p. 231]):

- There is a family of abelian groups (G_x) $(x \in Z^p \setminus Z^{p+1})$ and an isomorphism $\mathcal{G} \;\cong\; \oplus\, i_xG_x$.
- The natural maps $\mathcal{G} \leftarrow \varGamma_{Z^p}\mathcal{G} \to H^0_{Z^p/Z^{p+1}}\mathcal{G}$ are isomorphisms.

Thus for $\mathcal{F}^\bullet$ as above, the sheaf $H^n_{Z^p/Z^{p+1}}\mathcal{F}^\bullet$ lies on the Z^p/Z^{p+1}–skeleton of X.

A complex of abelian sheaves $\mathcal{G}^\bullet$ on X is called *a* $Z^\bullet$*-Cousin complex* if for each $p \in \mathbb{Z}$, $\mathcal{G}^p$ lies on the Z^p/Z^{p+1}–skeleton of X. The reference to $Z^\bullet$ is dropped in case of no ambiguity. Since $Z^p = X$ for some p, a Cousin complex is necessarily bounded below. The individual graded pieces $\mathcal{G}^p$, being direct sums of flasque sheaves, are themselves flasque.

By definition, the underlying graded object of a Cousin complex $\mathcal{G}^\bullet$ on X admits a decomposition parametrized by the points of X, and this decomposition is *finer* than the usual $\mathbb{Z}$-graded decomposition of $\mathcal{G}^\bullet$ as a complex. Specifically, one can associate to each $x \in X$ the abelian group $\mathcal{G}^\bullet(x)$ underlying $\varGamma_x\mathcal{G}^\bullet$; and since for any group G, $\varGamma_x(i_yG) = 0$ whenever $x \neq y$, the complex $\varGamma_x\mathcal{G}^\bullet$ vanishes in all degrees other than that $p = p(x)$ such that $x \in Z^p \setminus Z^{p+1}$, so $\mathcal{G}^\bullet(x) = \varGamma_x\mathcal{G}^{p(x)}$. In particular, there is a canonical isomorphism

$$\phi\colon \mathcal{G}^p \;\xrightarrow{\;\sim\;}\; \bigoplus_{x\in Z^p\setminus Z^{p+1}} i_x(\mathcal{G}^\bullet(x)),$$

uniquely determined by the property that for any $z \in Z^p \setminus Z^{p+1}$, the natural map $\mathcal{G}^\bullet(z) = \varGamma_z\mathcal{G}^p \xrightarrow{\varGamma_z(\phi)} \mathcal{G}^\bullet(z)$ is the identity map.

Let $\mathcal{G}^\bullet$ be a Cousin complex of abelian sheaves on X, and let $\mathcal{O}_X$ be a sheaf of commutative rings on X. If each $\mathcal{G}^\bullet(x)$ has an $\mathcal{O}_{X,x}$-module structure then all

[4] The necessary $\varGamma$-acyclicity properties of flasque sheaves are given in [**11**, Chap. IV, §1].

the component sheaves $\mathcal{G}^p$ have natural $\mathcal{O}_X$-module structures. Thus specifying a Cousin complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules with $\mathcal{O}_X$-linear differentials is equivalent to specifying the following data:

- for each $x \in X$, an $\mathcal{O}_{X,x}$-module $\mathcal{G}^\bullet(x)$;
- for each immediate specialization $x \rightsquigarrow y$ in X, an $\mathcal{O}_{X,y}$-linear map $\delta_{x,y} \colon \mathcal{G}^\bullet(x) \to \mathcal{G}^\bullet(y)$;

subject to the following conditions:

- for each $\xi \in \mathcal{G}^\bullet(x)$, $\delta_{x,y}(\xi) = 0$ for all but finitely many y;
- if $x \rightsquigarrow z$ is a specialization with $x \in Z^p$ and $z \in Z^{p+2}$, then $\Sigma_y\ \delta_{y,z}\delta_{x,y} = 0$ where y ranges over the set of all intermediate specializations $x \rightsquigarrow y \rightsquigarrow z$ with $x \neq y \neq z$.

Let X, $\mathcal{O}_X$ and $Z^\bullet$ be as before. To every $\mathcal{F}^\bullet \in \mathbf{D}^+(X)$ one associates functorially a Cousin $\mathcal{O}_X$-complex $E_{Z^\bullet}\mathcal{F}^\bullet$, as follows. Let $\mathcal{L}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. From the natural exact sequence

$$0 \to \mathit{\Gamma}_{Z^{p+1}}/\mathit{\Gamma}_{Z^{p+2}}\mathcal{L}^\bullet \to \mathit{\Gamma}_{Z^p}/\mathit{\Gamma}_{Z^{p+2}}\mathcal{L}^\bullet \to \mathit{\Gamma}_{Z^p}/\mathit{\Gamma}_{Z^{p+2}}\mathcal{L}^\bullet \to 0 \qquad (p \in \mathbb{Z}),$$

one derives connecting homology homomorphisms (see §1.4, (ix))

$$\delta^p \colon H^p_{Z^p/Z^{p+1}}\mathcal{F}^\bullet \longrightarrow H^{p+1}_{Z^{p+1}/Z^{p+2}}\mathcal{F}^\bullet,$$

whence the sequence

$$(9) \qquad \cdots \longrightarrow H^{p-1}_{Z^{p-1}/Z^p}\mathcal{F}^\bullet \xrightarrow{\ \delta^{p-1}\ } H^p_{Z^p/Z^{p+1}}\mathcal{F}^\bullet \xrightarrow{\ \delta^p\ } H^{p+1}_{Z^{p+1}/Z^{p+2}}\mathcal{F}^\bullet \longrightarrow \cdots$$

which is in fact a complex, denoted $E_{Z^\bullet}\mathcal{F}^\bullet$, with degree-$p$ component $H^p_{Z^p/Z^{p+1}}\mathcal{F}^\bullet$. (The filtration $\Gamma^\bullet$ of the identity functor gives rise to a spectral sequence whose $E_1^{p,0}$ terms form the complex $E_{Z^\bullet}\mathcal{F}^\bullet$. The spectral sequence converges to the homology of $\mathcal{F}^\bullet$ when X has finite Krull dimension, see [**11**, p. 227, p. 241].) The isomorphism in (8′) shows that $E_{Z^\bullet}\mathcal{F}^\bullet$ is a Cousin complex; and if $x \in Z^p \setminus Z^{p+1}$ then there are natural isomorphisms

$$(10) \qquad (E_{Z^\bullet}\mathcal{F}^\bullet)(x) = \mathit{\Gamma}_x(E_{Z^\bullet}\mathcal{F}^\bullet)^p \underset{(8')}{\xrightarrow{\ \sim\ }} \mathit{\Gamma}_x\Big(\bigoplus_{y \in Z^p\setminus Z^{p+1}} i_y(H^p_y\mathcal{F}^\bullet)\Big) \xrightarrow{\ \sim\ } H^p_x\mathcal{F}^\bullet.$$

Here are some basic properties of Cousin complexes, to be used in the next subsection.

Lemma 3.2.1. *Let X, $\mathcal{O}_X$ and $Z^\bullet$ be as before. Set $E := E_{Z^\bullet}$. Let $\mathcal{C}^\bullet$ be a Cousin $\mathcal{O}_X$-complex.*

(i) [a] *For any $m \geq n, p \in \mathbb{Z}$ and $x \in Z^p$ the natural maps are isomorphisms*

$$\sigma_{\geq n}\mathcal{C}^\bullet = \mathit{\Gamma}_{Z^n}\mathcal{C}^\bullet \xrightarrow{\ \sim\ } \mathbf{R}\mathit{\Gamma}_{Z^n}\mathcal{C}^\bullet,$$
$$\sigma_{\geq n}\sigma_{<m}\mathcal{C}^\bullet = \mathit{\Gamma}_{Z^n/Z^m}\mathcal{C}^\bullet \xrightarrow{\ \sim\ } \mathbf{R}\mathit{\Gamma}_{Z^n/Z^m}\mathcal{C}^\bullet,$$
$$\mathcal{C}^\bullet(x)[-p] = \mathit{\Gamma}_x\mathcal{C}^\bullet \xrightarrow{\ \sim\ } \mathbf{R}\mathit{\Gamma}_x\mathcal{C}^\bullet.$$

There result natural isomorphisms:

[b] $H^p\mathbf{R}\mathit{\Gamma}_x\mathcal{C}^\bullet = H^p_x\mathcal{C}^\bullet \xrightarrow{\ \sim\ } \mathcal{C}^\bullet(x) = H^p(\mathcal{C}^\bullet(x)[-p])$;

[c] $H^p_xE\mathcal{F}^\bullet \underset{[b]}{\xrightarrow{\ \sim\ }} (E\mathcal{F}^\bullet)(x) \underset{(10)}{\xrightarrow{\ \sim\ }} H^p_x\mathcal{F}^\bullet \qquad (\mathcal{F}^\bullet \in \mathbf{D}^+(X))$.

(ii) *The graded isomorphism* $E\mathcal{C}^\bullet \xrightarrow{\sim} \mathcal{C}^\bullet$ *made up of the punctual maps*

$$(E\mathcal{C}^\bullet)(x) = \varGamma_{\!x} H^p\mathbf{R}\varGamma_{Z^p/Z^{p+1}}\mathcal{C}^\bullet \underset{[\mathrm{a}]}{\xrightarrow{\sim}} \varGamma_{\!x} H^p\varGamma_{Z^p/Z^{p+1}}\mathcal{C}^\bullet = \varGamma_{\!x}\mathcal{C}^p = \mathcal{C}^\bullet(x)$$

is an isomorphism of complexes.

(iii) *The punctual isomorphism in* (ii) *factors as*

$$(E\mathcal{C}^\bullet)(x) \underset{(10)}{\xrightarrow{\sim}} H^p_x\mathcal{C}^\bullet \underset{(\mathrm{i})[\mathrm{b}]}{\xrightarrow{\sim}} \mathcal{C}^\bullet(x);$$

and hence the following diagram, with $\mathcal{F}^\bullet$, x *and* p *as in* (i)[c], *commutes*:

$$\begin{tikzcd}
(EE\mathcal{F}^\bullet)(x) \arrow[r, "(\mathrm{ii})", "\mathcal{C}^\bullet = E\mathcal{F}^\bullet"'] \arrow[d, "(10)"', "\simeq"] & (E\mathcal{F}^\bullet)(x) \arrow[d, "\simeq"', "(10)"] \\
H^p_xE\mathcal{F}^\bullet \arrow[r, "(\mathrm{i})[\mathrm{c}]"'] & H^p_x\mathcal{F}^\bullet
\end{tikzcd}$$

Proof. For any $Z' \subset Z \subset X$, flasque sheaves are acyclic for the functors $\varGamma_Z$ and $\varGamma_{Z/Z'}$ ([**25**, p. 37]). Since a Cousin complex consists of flasque sheaves, (i) results.

For (ii), we note first that in the above definition of $E\mathcal{F}^\bullet$ we can replace $\mathcal{L}^\bullet$ by any flasque resolution of $\mathcal{F}^\bullet$; so when $\mathcal{F}^\bullet$ is a Cousin complex $\mathcal{C}^\bullet$, we can take $\mathcal{L}^\bullet$ to be $\mathcal{C}^\bullet$ itself.

For any integer p, there is a natural commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varGamma_{Z^{p+1}/Z^{p+2}}\mathcal{C}^\bullet \arrow[r] \arrow[d, equal] & \varGamma_{Z^p/Z^{p+2}}\mathcal{C}^\bullet \arrow[r] \arrow[d, equal] & \varGamma_{Z^p/Z^{p+1}}\mathcal{C}^\bullet \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{C}^{p+1}[-p-1] \arrow[r] & \sigma_{\ge p}\sigma_{\le p+1}\mathcal{C}^\bullet \arrow[r] & \mathcal{C}^p[-p] \arrow[r] & 0
\end{tikzcd}$$

An examination of the p-th connecting homology homomorphisms derived from the rows (see (9)) makes (ii) clear.

With $\mathcal{C}^\bullet = E\mathcal{F}^\bullet$, the second assertion in (iii) amounts to commutativity—given by the first assertion—of the square in the following diagram.

$$\begin{tikzcd}
(E\mathcal{C}^\bullet)(x) \arrow[r, "(\mathrm{ii})"] \arrow[d, "(10)"] & \mathcal{C}^\bullet(x) \arrow[d, equal] & \\
H^p_x\mathcal{C}^\bullet \arrow[r, "(\mathrm{i})[\mathrm{b}]"] & \mathcal{C}^\bullet(x) \arrow[r, "(10)"] & H^p_x\mathcal{F}^\bullet
\end{tikzcd}$$

For the first assertion, we can use $\mathcal{C}^\bullet$ instead of $\mathcal{L}^\bullet$ to describe derived functors (see above). Then the definitions of the maps involved tell us that what has to be verified is commutativity of the following diagram (3.2.1.1) of natural isomorphisms, where the direct sums are all extended over $y \in Z^p \setminus Z^{p+1}$. But in that diagram, commutativity of subdiagram $\square_1$ follows from the description of the map in (8); and commutativity of the remaining subdiagrams is straightforward to check. $\square$

Let $(\mathcal{X}, \Delta) \in \mathbb{F}_{\mathrm{c}}$ (see §2.1). Then the codimension function Δ determines a filtration $Z^\bullet$ of $\mathcal{X}$ with

$$Z^p = \{x \in \mathcal{X} \mid \Delta(x) \ge p\}.$$

This filtration satisfies the properties (a)-(d) listed near the beginning of §3.2. (Property (a) is satisfied because Δ is bounded below by the least of its values at the generic points of the finitely many irreducible components of $\mathcal{X}$.) For this $Z^\bullet$,

$$
\begin{tikzcd}
\varGamma_{\!x} H^p \varGamma_{Z^p/Z^{p+1}}\mathcal{C}^\bullet \arrow[r] \arrow[dd] \arrow[dr, phantom, "\square_1"] & \varGamma_{\!x} H^p\big(\mathcal{C}^p[-p]\big) \arrow[r] \arrow[d] & \varGamma_{\!x}\mathcal{C}^p \arrow[d] \\
 & \varGamma_{\!x} H^p\big(\big(\oplus\, i_y\mathcal{C}^\bullet(y)\big)[-p]\big) \arrow[r] \arrow[d] & \varGamma_{\!x}\big(\oplus\, i_y\mathcal{C}^\bullet(y)\big) \arrow[dd, equal] \\
\varGamma_{\!x} H^p\big(\oplus\, i_y\varGamma_{\!y}\mathcal{C}^\bullet\big) \arrow[r] \arrow[d] & \varGamma_{\!x} H^p\big(\oplus\,\big(i_y\mathcal{C}^\bullet(y)[-p]\big)\big) \arrow[d] & \\
\varGamma_{\!x}\big(\oplus\, i_y H^p\varGamma_{\!y}\mathcal{C}^\bullet\big) \arrow[r] \arrow[d] & \varGamma_{\!x}\big(\oplus\, i_y H^p\big(\mathcal{C}^\bullet(y)[-p]\big)\big) \arrow[r] \arrow[d] & \varGamma_{\!x}\big(\oplus\, i_y\mathcal{C}^\bullet(y)\big) \arrow[d] \\
H^p\varGamma_{\!x}\mathcal{C}^\bullet \arrow[r] & H^p\big(\mathcal{C}^\bullet(x)[-p]\big) \arrow[r] & \mathcal{C}^\bullet(x)
\end{tikzcd}
$$

(3.2.1.1)

a $Z^\bullet$-Cousin complex is called a Δ-Cousin complex, and the Cousin functor $E_{Z^\bullet}$ is denoted by E_Δ. The relation of E_Δ to translation is immediate from the definition of $E_{Z^\bullet}$: for any complex $\mathcal{F}^\bullet$ and integer n,

$$E_{\Delta-n}\big(\mathcal{F}^\bullet[n]\big) = \big(E_\Delta\mathcal{F}^\bullet\big)[n]. \tag{11}$$

Our main interest is in the full subcategory $\mathrm{Coz}_\Delta(\mathfrak{X}) \subset \mathbf{C}(\mathfrak{X})$ whose objects are those Δ-Cousin complexes whose underlying modules are in $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$.

Let $(\mathfrak{X},\Delta)\in\mathbb{F}_{\mathrm{c}}$, $x\in\mathfrak{X}$, $\mathcal{F}^\bullet\in\mathbf{D}^+_{\mathrm{qc}}(\mathfrak{X})$, $\mathcal{G}^\bullet := E_\Delta\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F}^\bullet$. Then the composition

$$\mathcal{G}^\bullet(x) \xrightarrow{(10)} H^p_x\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F}^\bullet \xrightarrow[\text{(see §3.1)}]{(6)} H^p_{m_x}\mathcal{F}^\bullet_x$$

is a natural isomorphism

$$\mathcal{G}^\bullet(x) \xrightarrow{\ \sim\ } H^p_{m_x}\mathcal{F}^\bullet_x \qquad (p := \Delta(x)). \tag{12}$$

From 2.3.5 it follows that $\mathcal{G}^\bullet\in\mathrm{Coz}_\Delta(\mathfrak{X})$.

LEMMA 3.2.2. *Let $f\colon(\mathfrak{X},\Delta_1)\to(\mathfrak{Y},\Delta_2)$ be a flat map in $\mathbb{F}_{\mathrm{c}}$, $\mathcal{M}^\bullet$ a complex in $\mathrm{Coz}_{\Delta_2}(\mathfrak{Y})$ and $\mathcal{P}^\bullet$ a flat quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module. Let $x\in\mathfrak{X}$ and $q := \Delta_2(f(x))$. Then for any $i\in\mathbb{Z}$, with σ the truncation functor of* §1.4 (x), *and $\otimes := \otimes_{\mathcal{O}_{\mathfrak{X}}}$, the natural maps are isomorphisms*

$$
\begin{gathered}
H^i_{m_x}(f^*\sigma_{\ge q}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)_x \xrightarrow{\ \sim\ } H^i_{m_x}(f^*\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)_x \xrightarrow{\ \sim\ } H^i_{m_x}(f^*\sigma_{\le q}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)_x\,,\\
H^i_x\mathbf{R}\varGamma'_{\mathfrak{X}}(f^*\sigma_{\ge q}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet) \xrightarrow{\ \sim\ } H^i_x\mathbf{R}\varGamma'_{\mathfrak{X}}(f^*\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet) \xrightarrow{\ \sim\ } H^i_x\mathbf{R}\varGamma'_{\mathfrak{X}}(f^*\sigma_{\le q}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet).
\end{gathered}
$$

PROOF. The first two maps are isomorphic, respectively, to the last two, see (6), so it suffices to show that the last two are isomorphisms. Since f and $\mathcal{P}^\bullet$ are flat, the natural exact sequences

$$
\begin{gathered}
0\to\sigma_{\ge q}\mathcal{M}^\bullet\to\mathcal{M}^\bullet\to\sigma_{\le q-1}\mathcal{M}^\bullet\to 0,\\
0\to\sigma_{\ge q+1}\mathcal{M}^\bullet\to\mathcal{M}^\bullet\to\sigma_{\le q}\mathcal{M}^\bullet\to 0.
\end{gathered}
$$

induce exact sequences

$$
\begin{gathered}
0\to f^*\sigma_{\ge q}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet\to f^*\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet\to f^*\sigma_{\le q-1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet\to 0,\\
0\to f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet\to f^*\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet\to f^*\sigma_{\le q}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet\to 0.
\end{gathered}
$$

Applying the derived functors $\mathbf{R}\varGamma_x\mathbf{R}\varGamma'_{\mathcal{X}}$ to the triangles in $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ arising from the last two exact sequences, we get triangles in the derived category of complexes of $\mathcal{O}_{\mathcal{X},x}$-modules. The resulting long exact cohomology sequences show that for the lemma to hold it suffices that for all i,

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\le q-1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet) \;=\; 0 \;=\; H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet). \tag{13}$$

But (13) follows by induction from 3.1.11. For example, in the case of $\sigma_{\ge q+1}\mathcal{M}^\bullet$ we have for any $n\ge 1$ an exact sequence

$$0\to\sigma_{\ge q+n+1}\mathcal{M}^\bullet\to\sigma_{\ge q+n}\mathcal{M}^\bullet\to\mathcal{M}^{q+n}[-q-n]\to 0. \tag{14}$$

Since $\mathcal{M}^{q+n}=\oplus_y\, i_y\mathcal{M}^\bullet(y)$ where $y\ne f(x)$ (since $\Delta_2(y)\ne q$), 3.1.11 yields

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^{q+n}[-q-n]\otimes\mathcal{P}^\bullet)=0 \qquad \forall\, i.$$

Therefore from the long exact sequence corresponding to (14) we conclude that

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+n+1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)\cong H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+n}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet).$$

Hence by induction on n we see that for all $n\ge 1$ and for all i,

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)\cong H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+n}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet).$$

But for a fixed i and for n sufficiently large, the right hand side is 0, and thus

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)=0 \qquad \forall\, i.$$

A similar argument works for $\sigma_{\le q-1}\mathcal{M}^\bullet$. □

Applying 3.2.2 with $\sigma_{\le q}\mathcal{M}^\bullet$ in place of $\mathcal{M}^\bullet$, and using 3.1.11, we conclude:

Corollary 3.2.3. *In the situation of* 3.2.2 *there are natural isomorphisms*

$$\begin{array}{ccccc}
H^i_{m_x}(f^*\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet)_x & \xrightarrow{\ \sim\ } & H^i_{m_x}(f^*\mathcal{M}^q[-q]\otimes\mathcal{P}^\bullet)_x & \xrightarrow{\ \sim\ } & H^{i-q}_{m_x}(\mathcal{M}^\bullet(y)\otimes_{\mathcal{O}_{\mathcal{Y},y}}\mathcal{P}^\bullet_x)\\
{\scriptstyle(6)}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle(6)}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle(6)}\\
H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet\otimes\mathcal{P}^\bullet) & \xrightarrow{\ \sim\ } & H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^q[-q]\otimes\mathcal{P}^\bullet) & \xrightarrow{\ \sim\ } & H^{i-q}_x\mathbf{R}\varGamma'_{\mathcal{X}}(f^*i_y\mathcal{M}^\bullet(y)\otimes\mathcal{P}^\bullet)
\end{array}$$

3.3. Cohen-Macaulay complexes. Let $(\mathcal{X},\Delta)\in\mathbb{F}_{\mathrm{c}}$ and let $Z^\bullet$ denote the filtration of $\mathcal{X}$ induced by Δ, i.e., $Z^p=\{x\in\mathcal{X}\mid\Delta(x)\ge p\}$. We say that a complex $\mathcal{F}^\bullet\in\mathbf{D}^+(\mathcal{X})$ is *Cohen-Macaulay with respect to* Δ (in short Δ-CM, or simply CM) if it satisfies the following equivalent conditions (cf. [**11**, p. 242], (i)$\Leftrightarrow$(ii), where the boundedness assumption on $\mathcal{F}^\bullet$ is not used):

- (i) (a) For all integers $i<p$, $H^i_{Z^p}\mathcal{F}^\bullet=0$;
  - (b) For integers $i\ge p$ the canonical map $H^i_{Z^p}\mathcal{F}^\bullet\to H^i\mathcal{F}^\bullet$ is surjective when $i=p$ and bijective for $i>p$ (equivalently, $H^i_{X/Z^p}\mathcal{F}^\bullet=0$ for $i\ge p$);
- (ii) For any $x\in\mathcal{X}$, $H^i_x\mathcal{F}^\bullet=0$ for $i\ne\Delta(x)$ (equivalently, for any integers $i\ne p$, $H^i_{Z^p/Z^{p+1}}\mathcal{F}^\bullet=0$).

Let $\mathbf{D}^+(\mathcal{X};\Delta)_{\mathrm{CM}}$ be the full subcategory of $\mathbf{D}^+(\mathcal{X})$ whose objects are Δ-CM complexes. We now recall the relation between Δ-CM complexes and Δ-Cousin complexes. For the next few results we set aside considerations of quasi-coherence and torsionness. For convenience, we suppress the reference to the codimension function Δ in our notation. Thus $E=E_\Delta$ and $\mathbf{D}^+(\mathcal{X})_{\mathrm{CM}}=\mathbf{D}^+(\mathcal{X};\Delta)_{\mathrm{CM}}$.

Let $\mathrm{Cou}(\mathcal{X}) \subset \mathbf{C}(\mathcal{X})$ be the full subcategory of Cousin complexes. A basic property of $\mathrm{Cou}(\mathcal{X})$ is that if two maps h, k from $\mathcal{F}^\bullet$ to $\mathcal{G}^\bullet$ in $\mathrm{Cou}(\mathcal{X})$ are homotopy equivalent, then $h = k$. (More generally, for integers $p > q$ any map from $\mathcal{F}^p$ to $\mathcal{G}^q$ is zero since for any $x, y \in \mathcal{X}$ and abelian groups F, G, we have $\mathrm{Hom}(i_x F, i_y G) = 0$ if $y \notin \overline{\{x\}}$.) So $\mathrm{Cou}(\mathcal{X})$ can be considered as a full subcategory of $\mathbf{K}^+(\mathcal{X})$. Then *the localization functor* $Q\colon \mathbf{K}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$ *takes* $\mathrm{Cou}(\mathcal{X})$ *into* $\mathbf{D}^+(\mathcal{X})_{\mathrm{CM}}$. Indeed, by (i) of 3.2.1, for any $\mathcal{C}^\bullet \in \mathrm{Cou}(\mathcal{X})$ and $x \in \mathcal{X}$, with $p := \Delta(x)$ there is an isomorphism $H^j_x\mathcal{C}^\bullet \cong H^j(\mathcal{C}^\bullet(x)[-p])$ and therefore $H^j_x\mathcal{C}^\bullet = 0$ if $j \neq p$. Thus Q induces an additive functor

$$Q\colon \mathrm{Cou}(\mathcal{X}) \to \mathbf{D}^+(\mathcal{X})_{\mathrm{CM}}.$$

Let us elaborate. With

$$\overline{\mathbf{C}} := \mathrm{Cou}(\mathcal{X}), \qquad \overline{\mathbf{D}} := \mathbf{D}^+(\mathcal{X})_{\mathrm{CM}},$$

we shall assume for the rest of this subsection that Q (resp. E) is a functor from $\overline{\mathbf{C}}$ to $\overline{\mathbf{D}}$ (resp. $\overline{\mathbf{D}}$ to $\overline{\mathbf{C}}$).

PROPOSITION 3.3.1. *With preceding notation, the functor* $Q\colon \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ *is an* equivalence of categories *having E as a pseudo-inverse equivalence. In particular, E is fully faithful.*

PROOF. That Q is an equivalence of categories is one of the main results (Theorem 3.9) in [**25**]. Let $S\colon \overline{\mathbf{D}} \to \overline{\mathbf{C}}$ be a pseudo-inverse equivalence. Since $EQ \xrightarrow{\sim} 1_{\overline{\mathbf{C}}}$ (3.2.1(ii)), there are isomorphisms $E \xrightarrow{\sim} EQS) \xrightarrow{\sim} S$. Thus E is a pseudo-inverse of Q. $\square$

COROLLARY 3.3.2. *Let* $EQ \xrightarrow{\sim} 1_{\overline{\mathbf{C}}}$ *be the isomorphism of* 3.2.1(ii). *Then there exists a unique isomorphism of functors* $\mathsf{S} = \mathsf{S}_{\mathcal{X},\Delta}\colon 1_{\overline{\mathbf{D}}} \xrightarrow{\sim} QE$ *such that the following two induced isomorphisms are equal*:

$$E = E1_{\overline{\mathbf{D}}} \xrightarrow{\sim} EQE, \qquad\qquad E = 1_{\overline{\mathbf{C}}}E \xrightarrow{\sim} EQE.$$

Furthermore, the following induced isomorphisms are equal:

$$QEQ \xrightarrow{\sim} Q1_{\overline{\mathbf{C}}} = Q, \qquad\qquad QEQ \xrightarrow{\sim} 1_{\overline{\mathbf{D}}}Q = Q.$$

PROOF. Since E is fully faithful, for any $\mathcal{F}^\bullet \in \overline{\mathbf{D}}$ the map $E\mathcal{F}^\bullet \xrightarrow{\sim} EQE\mathcal{F}^\bullet$ given by $E = 1_{\overline{\mathbf{C}}}E \xrightarrow{\sim} EQE$ comes via E from a unique functorial isomorphism $\mathcal{F}^\bullet \xrightarrow{\sim} QE\mathcal{F}^\bullet$ that fulfills the first assertion. The second assertion need only be shown after application of the functor E, at which point the first assertion reduces the problem to showing that the following isomorphisms are equal

$$EQEQ \xrightarrow{\sim} EQ1_{\overline{\mathbf{C}}} = EQ, \qquad\qquad EQEQ \xrightarrow{\sim} 1_{\overline{\mathbf{C}}}EQ = EQ,$$

which is easy to do after composition with the isomorphism $EQ \xrightarrow{\sim} 1_{\overline{\mathbf{C}}}$. $\square$

COROLLARY 3.3.3. *For any* $\mathcal{F}^\bullet \in \overline{\mathbf{D}}$ *the isomorphism* $\mathsf{S}(\mathcal{F}^\bullet)\colon \mathcal{F}^\bullet \xrightarrow{\sim} QE\mathcal{F}^\bullet$ *obtained in* 3.3.2 *is the unique one satisfying the property that for any* $x \in \mathcal{X}$, *with* $p := \Delta(x)$, *the induced isomorphism* $H^p_x\,\mathsf{S}(\mathcal{F}^\bullet)\colon H^p_x\mathcal{F}^\bullet \xrightarrow{\sim} H^p_xQE\mathcal{F}^\bullet$ *is the inverse of the isomorphism in* 3.2.1(i)[c].

Proof. Let $\phi\colon \mathcal{F}^\bullet \xrightarrow{\sim} QE\mathcal{F}^\bullet$ be an isomorphism. Consider the following diagram, where the vertical isomorphisms are given by the punctual decomposition of $E(-)$ in (10):

$$\begin{tikzcd}
(E\mathcal{F}^\bullet)(x) \arrow[r, "(E\phi)(x)"] \arrow[d, "\cong"'] & (EQE\mathcal{F}^\bullet)(x) \arrow[r, "EQ \cong 1_{\overline{\mathbf{C}}}"] \arrow[d] & (E\mathcal{F}^\bullet)(x) \arrow[d, "\cong"] \\
H^p_x\mathcal{F}^\bullet \arrow[r, "H^p_x\phi"'] & H^p_xQE\mathcal{F}^\bullet \arrow[r, "3.2.1(\mathrm{i})[\mathrm{c}]"'] & H^p_x\mathcal{F}^\bullet
\end{tikzcd}$$

The rectangle on the left commutes for functorial reasons, while the one on the right commutes by 3.2.1(iii). If $\phi = \mathsf{S}(\mathcal{F}^\bullet)$, then by 3.3.2, the top row composes to an identity map, and hence so does the bottom. Conversely, if the bottom row composes to an identity map, then so does the top. Since this holds for every $x \in \mathcal{X}$, therefore the composite map $E\mathcal{F}^\bullet \xrightarrow{E\phi} EQE\mathcal{F}^\bullet \xrightarrow[3.2.1(\mathrm{ii})]{EQ \xrightarrow{\sim} 1_{\overline{\mathbf{C}}}} E\mathcal{F}^\bullet$ is identity and hence $\phi = \mathsf{S}(\mathcal{F}^\bullet)$. $\square$

Let $x \in \mathcal{X}$ and $p := \Delta(x)$. Any $\mathcal{O}_\mathcal{X}$-complex $\mathcal{C}^\bullet$, has the subcomplex $\mathit{\Gamma}_{\overline{\{x\}}}\mathcal{C}^\bullet$, and for the stalks at x there is the inclusion map $\gamma = \gamma(x, \mathcal{C}^\bullet)\colon \mathit{\Gamma}_x\mathcal{C}^\bullet \hookrightarrow \mathcal{C}^\bullet_x$. Since "stalk at x" is an exact functor, $Q\gamma$ factors naturally as

$$\mathit{\Gamma}_x\mathcal{C}^\bullet \to \mathbf{R}\mathit{\Gamma}_x\mathcal{C}^\bullet \xrightarrow{\bar\gamma} \mathcal{C}^\bullet_x$$

with $\bar\gamma = \bar\gamma(x, \mathcal{C}^\bullet)$ a $\mathbf{D}(\mathcal{O}_{\mathcal{X},x})$-morphism. So for $\mathcal{F}^\bullet \in \mathbf{D}^+(\mathcal{X})$ there is a natural map

$$\eta_1 = \eta_1(x, \mathcal{F}^\bullet)\colon \big(H^p_x\mathcal{F}^\bullet\big)[-p] \underset{(10)}{\xrightarrow{\sim}} (E\mathcal{F}^\bullet)(x)[-p] = \mathit{\Gamma}_x E\mathcal{F}^\bullet \xrightarrow{\gamma} \big(E\mathcal{F}^\bullet\big)_x.$$

If $\mathcal{F}^\bullet \in \overline{\mathbf{D}}$ then by definition, $H^i_x\mathcal{F}^\bullet = 0$ for $i \neq p$. Hence the natural maps (see §1.4, (x)) are isomorphisms

$$\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet \xrightarrow{\sim} \tau_{\geq p}\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet \xleftarrow{\sim} \tau_{\leq p}\tau_{\geq p}\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet = \big(H^p_x\mathcal{F}^\bullet\big)[-p].$$

Let $\eta_2 = \eta_2(x, \mathcal{F}^\bullet)$ be the resulting isomorphism $\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet \xrightarrow{\sim} \big(H^p_x\mathcal{F}^\bullet\big)[-p]$.

Corollary 3.3.4. *For any $\mathcal{F}^\bullet \in \overline{\mathbf{D}}$, set $\mathcal{E}^\bullet := QE\mathcal{F}^\bullet$ and let $\nu\colon \mathcal{F}^\bullet \to \mathcal{E}^\bullet$ be the isomorphism given by* 3.3.2. *Then with notation as above the following diagram in $\mathbf{D}(\mathcal{O}_{\mathcal{X},x})$ commutes:*

$$\begin{tikzcd}
\mathcal{F}^\bullet_x \arrow[r, "\sim", "\nu_x"'] & \mathcal{E}^\bullet_x \\
\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet \arrow[u, "\bar\gamma"] \arrow[r, "\sim", "\eta_2"'] & \big(H^p_x\mathcal{F}^\bullet\big)[-p] \arrow[u, "\eta_1"']
\end{tikzcd}$$

Proof. Expand the diagram thus:

$$\begin{tikzcd}
\mathcal{F}^\bullet_x \arrow[r, "\sim", "\nu_x"'] & \mathcal{E}^\bullet_x \arrow[rr, equal] & & \big(E\mathcal{F}^\bullet\big)_x \\
\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet \arrow[u, "\bar\gamma"] \arrow[r, "\eta_2"'] \arrow[ur, phantom, "\square_1"] \arrow[d, equal] & \big(H^p_x\mathcal{F}^\bullet\big)[-p] \arrow[u, "\eta_1"'] \arrow[r, "\sim", "(10)"'] \arrow[urr, phantom, "\square_2"] & (E\mathcal{F}^\bullet)(x)[-p] \arrow[r, equal] \arrow[d, phantom, "\square_3"] & \mathit{\Gamma}_x(E\mathcal{F}^\bullet) \arrow[u, "\gamma_x"'] \\
\mathbf{R}\mathit{\Gamma}_x\mathcal{F}^\bullet \arrow[rrr, "\sim", "\mathbf{R}\mathit{\Gamma}_x\nu"'] & & & \mathbf{R}\mathit{\Gamma}_x\mathcal{E}^\bullet \arrow[u, "3.2.1", "\simeq"']
\end{tikzcd}$$

By definition, the rightmost column composes to $\bar\gamma(x,\mathcal{E}^\bullet)$, and so the outer border of this expanded diagram commutes.

Subdiagram $\square_2$ commutes by definition.

According to 3.3.3, application of the homology functor H^p to subdiagram $\square_3$ produces a commutative diagram. But that means $\square_3$ itself commutes, because its vertices are complexes which have vanishing homology in all degrees except p, and H^p is an equivalence from the full subcategory of such complexes in $\mathbf{D}(\mathcal{O}_{\mathcal{X},x})$ to the category of $\mathcal{O}_{\mathcal{X},x}$-modules (with pseudo-inverse $F \mapsto F[-p]$).

It results then that $\square_1$ commutes, as asserted. $\square$

4. Generalized fractions and pseudofunctors

Below we recall the definition of a pseudofunctor. In 4.3.1 we give a modified version of Huang's pseudofunctor constructed in [**13**]. Various components of this pseudofunctor have an explicit description in terms of generalized fractions. Generalized fractions are useful in denoting elements of certain local cohomology modules. In §4.1 we review the definition of generalized fractions and state the relation between two different competing definitions. In §4.2 we review some isomorphisms that are needed to describe the pseudofunctor discussed in 4.3.1. In §4.4 we prove a somewhat technical result which is mainly used in proving commutativity of the diagram in (43) in §5.4.

A *contravariant pseudofunctor* '$\#$' (or $(-)^{\#}$) on a category C assigns to each C-object X a category $X^{\#}$, to each C-morphism $f\colon X \to Y$ a contravariant functor $f^{\#}\colon Y^{\#} \to X^{\#}$, to each C-diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial *comparison isomorphism* $C^{\#}_{f,g}\colon f^{\#}g^{\#} \xrightarrow{\sim} (gf)^{\#}$, and to each C-object Z a functorial *unit isomorphism* $\delta^{\#}_{Z}\colon (1_Z)^{\#} \xrightarrow{\sim} 1_{Z^{\#}}$, all subject to the following conditions:

i) For every triple of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in C the following associativity diagram commutes.

$$\begin{array}{ccc} f^{\#}g^{\#}h^{\#} & \xrightarrow{f^{\#}C^{\#}_{g,h}} & f^{\#}(hg)^{\#} \\ {\scriptstyle C^{\#}_{f,g}h^{\#}}\big\downarrow & & \big\downarrow{\scriptstyle C^{\#}_{f,hg}} \\ (gf)^{\#}h^{\#} & \xrightarrow[C^{\#}_{gf,h}]{} & (hgf)^{\#} \end{array}$$

ii) For any map $f\colon X \to Y$ in C, the following two compositions are identity

$$(1_X)^{\#}f^{\#} \xrightarrow{C^{\#}_{1_X,f}} f^{\#} \xrightarrow{\delta^{\#\,-1}_{X}f^{\#}} (1_X)^{\#}f^{\#},$$

$$f^{\#}(1_Y)^{\#} \xrightarrow{C^{\#}_{f,1_Y}} f^{\#} \xrightarrow{f^{\#}\delta^{\#\,-1}_{Y}} f^{\#}(1_Y)^{\#}.$$

If necessary, we also use the cumbersome notation of a quadruple

$$\Big(\,(-)^{\#},\ (-)^{\#},\ C^{\#}_{(-),(-)},\ \delta^{\#}_{(-)}\,\Big)$$

to denote the pseudofunctor '$\#$' where the first entry operates on the objects of C, the second on morphisms, the third on pairs of composable morphisms and the fourth on objects of C, all having the obvious meanings as per the definition above.

A *covariant pseudofunctor* is defined by associating to each morphism in C a covariant functor which is required to satisfy appropriately modified compatibility conditions. We will use the subscript notation $(-)_{\#}$ when dealing with covariant pseudofunctors (cf. [**13**, chapter IV]).

Remark. The definition is simpler when for all Z, $(1_Z)^{\#} = 1_{Z^{\#}}$ and $\delta^{\#}_{Z}$ is the identity map. A pseudofunctor satisfying these additional conditions is said to be *normalized.* Replacing the functor $(1_Z)^{\#}$ by $1_{Z^{\#}}$ for every Z and replacing $C^{\#}_{f,g}$ by $C^{\#}_{gf}$ whenever f or g is an identity map transforms any pseudofunctor $(-)^{\#}$ into an isomorphic (in the obvious sense) normalized one.

A normalized contravariant pseudofunctor on C is the same thing as a contravariant 2-functor from C to the 2-category of all categories.

4.1. Generalized fractions. Let A be a noetherian ring and M an A-module. Let $\mathbf{t} = (t_1, \dots, t_n)$ be a sequence in A and let $\mathfrak{a}$ denote an ideal in A such that $\sqrt{\mathfrak{a}} = \sqrt{\mathbf{t}A}$. Then the elements of the local cohomology module $H^n_{\mathfrak{a}}M$ can be represented in the form of generalized fractions of the type

$$\begin{bmatrix} m \\ t_1^{b_1}, \dots, t_n^{b_n} \end{bmatrix}, \qquad m \in M, \qquad b_i > 0,$$

however there are two natural ways of doing so which we now review. We state the relationship between these two ways in Lemma 4.1.1.

The first method involves using the Čech complex. Let $X = \operatorname{Spec}(A)$, let Z denote the closed subset of X defined by the ideal $\mathfrak{a}$ and let U denote the complement $X \setminus Z$. Let $\mathfrak{U} = (U_1, \dots, U_n)$ denote the affine open covering of U associated to the sequence $\mathbf{t}$ where $U_i = \operatorname{Spec}(A_{t_i})$. Let $\check{C}^\bullet(\mathfrak{U}, M)$ denote the Čech complex of A-modules for the sheaf $M^\sim\big|_U$ (see 2.2.1), corresponding to the cover $\mathfrak{U}$ of U. Let $M \to I^\bullet$ denote an injective resolution of M. Then $M^\sim \to I^{\bullet\sim}$ is a flasque resolution, in fact a resolution by $\mathcal{A}_{\mathrm{qc}}(X)$-injectives. We may therefore make the identification $H^*(U, M^\sim) = H^*\Gamma(U, I^{\bullet\sim})$. The complex $\check{C}^\bullet(\mathfrak{U}, M)$ exists between degrees 0 and $n-1$ and we have a natural map

$$M_{t_1\cdots t_n} = \check{C}^{n-1}(\mathfrak{U}, M) \twoheadrightarrow H^{n-1}\check{C}^\bullet(\mathfrak{U}, M) \xrightarrow{\alpha} H^{n-1}(U, M^\sim) \xrightarrow{\eta} H^n_{\mathfrak{a}}M. \tag{15}$$

where α is the standard isomorphism relating the Čech cohomology to the usual one (see [**12**, III, Lemma 4.4]) and η is the usual connecting homomorphism in the homology long exact sequence associated to the exact sequence

$$0 \to \varGamma_{\mathfrak{a}} I^\bullet \to I^\bullet \to \varGamma\,(U, I^{\bullet\sim}) \to 0.$$

The map η is surjective for $n \geq 1$ and an isomorphism for $n > 1$. Therefore any element $\theta \in H^n_{\mathfrak{a}}M$ is the image of an element in $M_{t_1\cdots t_n}$. This leads to one way of representing a cohomology class by a generalized fraction, viz., for $\theta \in H^n_{\mathfrak{a}}M$, for $m \in M$ and for positive integers b_i we say that

$$\theta = \begin{bmatrix} m \\ t_1^{b_1}, \dots, t_n^{b_n} \end{bmatrix}_{\mathrm{C}}$$

(the subscript C stands for Čech) if the element

$$\frac{m}{t_1^{b_1} \cdots t_n^{b_n}} \in M_{t_1\cdots t_n}$$

goes to θ under the natural map of (15). We call such a generalized fraction a *C-fraction* representing θ.

The second method involves the direct-limit Koszul complexes. Let $K^\bullet_\infty(\mathbf{t})$ denote the direct-limit Koszul complex on $\mathbf{t}$ over A. With $I^\bullet$ denoting an injective resolution of M, there are natural quasi-isomorphisms (with $\otimes = \otimes_A$)

$$K^\bullet_\infty(\mathbf{t}) \otimes M \to K^\bullet_\infty(\mathbf{t}) \otimes I^\bullet \leftarrow \varGamma_{\mathbf{t}A} I^\bullet = \varGamma_{\mathfrak{a}} I^\bullet. \qquad \text{(cf. 3.1.4)}$$

Therefore we have a natural map

$$M_{t_1\cdots t_n} = K^n_\infty(\mathbf{t}) \otimes M \twoheadrightarrow H^n(K^\bullet_\infty(\mathbf{t}) \otimes M) \cong H^n_{\mathfrak{a}}M. \tag{16}$$

We represent any $\theta \in H^n_{\mathfrak{a}}M$ by a generalized fraction

$$\theta = \begin{bmatrix} m \\ t_1^{b_1}, \dots, t_n^{b_n} \end{bmatrix}_{\mathrm{K}}$$

(the subscript K being for Koszul) if the element

$$\frac{m}{t_1^{b_1}\cdots t_n^{b_n}} \in M_{t_1\cdots t_n}$$

goes to θ under the natural map of (16). We call such a generalized fraction a *K-fraction* representing θ.

LEMMA 4.1.1. *With $A, M, \mathbf{t}, n$ as above, for any $m \in M$ and integers $b_i > 0$ with i between 1 and n, the following relation holds.*

$$\begin{bmatrix} m \\ t_1^{b_1}, \dots, t_n^{b_n} \end{bmatrix}_K = (-1)^n \begin{bmatrix} m \\ t_1^{b_1}, \dots, t_n^{b_n} \end{bmatrix}_C$$

PROOF. Before we begin with the proof we will take a short digression regarding Čech resolutions. Let $X = \operatorname{Spec}(A)$, U, $\mathfrak{U}$ be as above. Let $\mathcal{O}_U \to \mathcal{C}^\bullet$ denote the corresponding Čech resolution of $\mathcal{O}_U$ on U. For an A-module N, set $N_U^\sim := N^\sim\big|_U$. We now recall some basic facts concerning $\mathcal{C}^\bullet$.

The complex $\mathcal{C}^\bullet$ consists of flat $\mathcal{O}_U$-modules. For any A-module N, the natural induced map $N_U^\sim \to \mathcal{C}^\bullet \otimes_U N_U^\sim$ can be canonically identified with the corresponding Čech resolution and if N is injective, so that $N_U^\sim$ consists of $\mathcal{A}_{\mathrm{qct}}(U)$-injectives then $\mathcal{C}^\bullet \otimes_U N_U^\sim$ also consists of $\mathcal{A}_{\mathrm{qct}}(U)$-injectives. Set $C^\bullet = \Gamma(U, \mathcal{C}^\bullet)$. Then $C^\bullet$ consists of flat A-modules and there is a natural isomorphism

$$C^\bullet \otimes_A N \xrightarrow{\ \sim\ } \Gamma(U, \mathcal{C}^\bullet \otimes_U N_U^\sim) = \check{C}^\bullet(\mathfrak{U}, N).$$

It follows that if $M \to I^\bullet$ is an injective resolution then in the following diagram having obvious natural maps,

$$\begin{tikzcd} M_U^\sim \arrow[r] \arrow[d] & \mathcal{C}^\bullet \otimes_U M_U^\sim \arrow[d, "\beta_1"] \\ I_U^{\bullet\sim} \arrow[r, "\beta_2"] & \mathcal{C}^\bullet \otimes_U I_U^{\bullet\sim} \end{tikzcd} \tag{17}$$

all the maps are quasi-isomorphisms and β_2 has a homotopy inverse. Moreover, there are isomorphisms

$$\begin{aligned} H^i\check{C}^\bullet(\mathfrak{U}, M) = H^i\Gamma(U, \mathcal{C}^\bullet \otimes_U M_U^\sim) &\xrightarrow[\text{via } \beta_1]{\sim} H^i\Gamma(U, \mathcal{C}^\bullet \otimes_U I_U^{\bullet\sim}) \\ &\xrightarrow[\text{via } \beta_2^{-1}]{\sim} H^i\Gamma(U, I_U^{\bullet\sim}) = H^i(U, M_U^\sim). \end{aligned}$$

It follows that the map α in (15) is obtained from this process. It then follows that the map $M_{t_1\cdots t_n} \twoheadrightarrow H^n_{\mathfrak{a}}M$ of (15) is also obtained from the sequence

$$\begin{aligned} M_{t_1\cdots t_n} = C^{n-1} \otimes_A M \twoheadrightarrow H^{n-1}(C^\bullet \otimes_A M) &\xrightarrow[\text{via } \beta_1]{\sim} H^{n-1}(C^\bullet \otimes_A I^\bullet) \\ &\xrightarrow[\text{via } \beta_2^{-1}]{\sim} H^{n-1}(I^{\bullet\sim}(U)) \xrightarrow{\eta} H^n\varGamma_{\mathfrak{a}} I^\bullet. \end{aligned}$$

Finally, we recall that $C^\bullet$ is a displaced (and truncated) version of the complex $K^\bullet := K^\bullet_\infty(\mathbf{t})$. More precisely, for all p except $p = -1$, we have $C^p = K^{p+1}$ and $d^p_{C^\bullet} = d^{p+1}_{K^\bullet}$ where d stands for the corresponding differentials. In particular, the graded maps $C^p \to K^{p+1}$, defined as $(-1)^{p+1}$ times the identity map, form a map of complexes $\psi\colon C^\bullet \to K^\bullet[1]$.

Returning to the proof of 4.1.1 consider the following diagram with maps described below (we use $\otimes = \otimes_A$ for the rest of the proof).

$$
\begin{tikzcd}
H^{n-1}(C^\bullet \otimes M) \arrow[r, "\mu_1"] \arrow[d, "\mu_3"] & H^{n-1}(K^\bullet[1] \otimes M) \arrow[r, "\mu_2"] \arrow[d, "\mu_4"] & H^n(K^\bullet \otimes M) \arrow[d, "\mu_5"] \\
H^{n-1}(C^\bullet \otimes I^\bullet) \arrow[r, "\mu_6"] & H^{n-1}(K^\bullet[1] \otimes I^\bullet) \arrow[r, "\mu_7"] & H^n(K^\bullet \otimes I^\bullet) \\
H^{n-1}(I^{\bullet\sim}(U)) \arrow[r, "\mu_{10}"'] \arrow[u, "\mu_8"] & H^n\varGamma_{\mathfrak a} I^\bullet \arrow[r, equal] & H^n\varGamma_{\mathfrak a} I^\bullet \arrow[u, "\mu_9"]
\end{tikzcd}
\tag{18}
$$

The maps μ_1, μ_6 are induced by $\psi \otimes 1$. while μ_2, μ_7 are induced by the natural isomorphism $K^\bullet[1]\otimes - \xrightarrow{\ \sim\ } (K^\bullet \otimes -)[1]$ obtained using the convention in §1.4, (iv). It follows that the underlying maps of complexes of μ_2, μ_7 are identity maps at the graded level. The maps μ_3, μ_4, μ_5 are the obvious natural ones. The map μ_8 is the isomorphism induced by β_2 in (17). In particular, the underlying map of complexes of μ_8 is given, at the graded level, by the restriction maps

$$I^{p\sim}(U) \longrightarrow \oplus_i \, I^p_{t_i} \;=\; C^0 \otimes I^p \subset (C^\bullet \otimes I^\bullet)^p.$$

The map μ_9 is the usual isomorphism (see 3.1.4) while μ_{10} is the connecting homomorphism referred to in the definition of η in (15).

It suffices to show that the diagram in (18) commutes. Assuming that (18) commutes, one proves the Lemma as follows. The composition $\mu_{10}\mu_8^{-1}\mu_3$ defines the map used in (15) while the composition $\mu_9^{-1}\mu_5$ defines the map used in (16). From the definitions involved we see that the map $\mu_2\mu_1$ sends the cohomology class of the cocycle $\frac{m}{t_1^{b_1}\cdots t_n^{b_n}} \in C^{n-1} \otimes M$ to the cohomology class of $(-1)^n \frac{m}{t_1^{b_1}\cdots t_n^{b_n}} \in K^n \otimes M$, thereby proving the Lemma.

In (18) the top two rectangles commute due to functorial reasons. To prove commutativity of the bottom part, consider an element $\zeta \in H^{n-1}(I^{\bullet\sim}(U))$. Since the natural map $I^{n-1} \to I^{n-1\sim}(U)$ is surjective (as $I^{n-1\sim}$ is flasque) there exists an element, say z, in I^{n-1} whose restriction to $I^{n-1\sim}(U)$ is a cocycle mapping to ζ. Let $\delta^\bullet$ denote the differential of the complex $I^\bullet$. Then $\delta^{n-1}z$ maps to zero in $I^{n\sim}(U)$ and so lies in $\varGamma_{\mathfrak a} I^n$ and is an n-cocycle in $\varGamma_{\mathfrak a} I^n$. From the definition of μ_{10} we see that $\mu_{10}(\zeta) = [\delta^{n-1}z]$. Under the natural map $\varGamma_{\mathfrak a} I^n \to I^n = K^0 \otimes I^n \subset (K^\bullet \otimes I^\bullet)^n$ the element $\delta^{n-1}z$ maps to $1 \otimes \delta^{n-1}z$ and therefore we have

$$\mu_9\mu_{10}(\zeta) = [1 \otimes \delta^{n-1}z].$$

Traveling the other route in the bottom half of (18), under the sequence of natural maps $I^n \twoheadrightarrow I^{n-1\sim}(U) \to \oplus_i I^{n-1}_{t_i} = C^0 \otimes I^{n-1} \subset (C^\bullet \otimes I^\bullet)^{n-1}$, the element z goes to $y := (z/1, \dots, z/1)$. Therefore $\mu_8(\zeta) = [y]$. Considering y as an element of $K^1 \otimes I^{n-1}$ via the identification $K^1 = C^0$, we see that

$$\mu_7\mu_6\mu_8(\zeta) = [-y].$$

To finish the proof we must verify that $\mu_9\mu_{10}(\zeta) = \mu_7\mu_6\mu_8(\zeta)$, which amounts verifying that the element $\beta = 1 \otimes \delta^{n-1}z \;+\; y$ in $(K^\bullet \otimes I^\bullet)^n$ is cohomologous to zero, i.e., β is a coboundary. The element $\gamma = 1 \otimes z \in K^0 \otimes I^n \subset (K^\bullet \otimes I^\bullet)^{n-1}$ maps to β. $\square$

From now on, unless specifically mentioned, all generalized fractions, by default are C-fractions.

As a final remark note that, with notation as above, if J is an ideal in A such that M is J-torsion, then the natural induced map $H^*_{\mathfrak{a}+J}M \to H^*_{\mathfrak{a}}M$ is an isomorphism. In particular, we may also represent the elements of $H^n_{\mathfrak{a}+J}M$ by generalized fractions.

4.2. Some local isomorphisms. We now describe some local isomorphisms that form the base of the construction of a pseudofunctor over local rings. By default, any local ring is considered a topological ring under the topology given by the powers of the maximal ideal. If any letter such as $R, S, A, B, \dots$ denotes a local ring then the corresponding maximal ideal is denoted by $m_R, m_S, m_A, m_B, \dots$.

Let $R \to S$ be a local homomorphism of noetherian local rings. Let M be an m_R-torsion R-module and L an S-module. Let $\widehat{R}, \widehat{S}$ denote the respective completions of R, S along the corresponding maximal ideals. Then M can naturally be considered as an $\widehat{R}$-module. Set $\widehat{L} := L \otimes_S \widehat{S}$. For any integer r, the S-modules $H^r_{m_S}(M \otimes_R L)$ and $\mathrm{Hom}^{\mathrm{c}}_R(S, M)$ (where $\mathrm{Hom}^{\mathrm{c}}_R$ denotes continuous Hom as R-modules) are m_S-torsion modules and hence naturally inherit the structure of an $\widehat{S}$-module. Moreover there are following isomorphisms.

For any integer r, there is a natural isomorphism

$$H^r_{m_S}(M \otimes_R L) \xrightarrow{\ \sim\ } H^r_{m_{\widehat{S}}}(M \otimes_{\widehat{R}} \widehat{L}_1) \tag{19}$$

defined by the following sequence of natural maps (with $m = m_S, \widehat{m} = \widehat{m_S}$)

$$H^r_m(M \otimes_R L) \xrightarrow{\alpha_1} (H^r_m(M \otimes_R L)) \otimes_S \widehat{S} \xrightarrow{\alpha_2} H^r_m((M \otimes_R L) \otimes_S \widehat{S})$$

$$\xrightarrow{\ \sim\ } H^r_m(M \otimes_R \widehat{L}_1) \xleftarrow{\alpha_3} H^r_m(M \otimes_{\widehat{R}} \widehat{L}_1) \xrightarrow{\alpha_4} H^r_{m\widehat{S}}(M \otimes_{\widehat{R}} \widehat{L}_1) = H^r_{\widehat{m}}(M \otimes_{\widehat{R}} \widehat{L}_1),$$

where α_i are seen to be isomorphisms due to the following reasons. For α_1 we use the fact that $H^r_{m_S}(M \otimes_R L)$ is m_S-torsion, for α_2 we use flatness of $\widehat{S}$ over S, for α_3 we use that M is m_R-torsion and finally α_4 is the isomorphism corresponding to extension of scalars.

The following natural map is an isomorphism

$$\mathrm{Hom}^{\mathrm{c}}_R(S, M) \longrightarrow \mathrm{Hom}^{\mathrm{c}}_{\widehat{R}}(\widehat{S}, M) \tag{20}$$

as is seen by taking direct limits over i of the following sequence

$$\mathrm{Hom}_R(S/m^i_S\,, M) \cong \mathrm{Hom}_{R/m^i_R}(\widehat{S}/m^i_{\widehat{S}}\,, M) \cong \mathrm{Hom}_{\widehat{R}/m^i_{\widehat{R}}}(\widehat{S}/m^i_{\widehat{S}}\,, M).$$

In the rest of this subsection we shall consider iterated versions of the isomorphisms in (19) and (20). These isomorphisms give the comparison maps of the pseudofunctor in 4.3.

4.2.1. Let $A \to B \to C$ be local homomorphisms of noetherian local rings. Let I be an ideal in B and N an I-torsion B-module. Let $i = \dim(B/I)$ and $j = \dim(C/m_B C)$. For any C-module L_2 there exists a canonical isomorphism

$$H^j_{m_C}(H^i_{m_B}(N) \otimes_B L_2) \xrightarrow{\ \sim\ } H^{i+j}_{m_C}(N \otimes_B L_2), \tag{21}$$

that is described in terms of generalized fractions as follows. Let $\mathbf{s}$ be a system of parameters of length i for m_B/I and let $\mathbf{t}$ be a system of parameters of length j

for m_C/m_BC. Then (21) assigns

$$\begin{bmatrix} \begin{bmatrix} n \\ \mathbf{s} \end{bmatrix} \otimes l \\ \mathbf{t} \end{bmatrix} \longrightarrow \begin{bmatrix} n \otimes l \\ \mathbf{s}, \mathbf{t} \end{bmatrix}, \qquad \{n \in N, l \in L_2\}. \tag{21$'$}$$

(By varying $\mathbf{s}, \mathbf{t}$ we see that (21′) uniquely determines (21). The existence and canonicity of (21) follows from [**13**, 2.5], recalled in (22*) below, modulo an isomorphism that reverses the order of the tensor product and a constant sign factor depending only on i, j. Also, see 4.4.1 for another definition.) Now suppose M is a zero-dimensional A-module and L_1 a B-module. Set $I := m_AB$. Then the module $N = M \otimes_A L_1$ is I-torsion and hence from (21) we obtain

$$H^j_{m_C}(H^i_{m_B}(M \otimes_A L_1) \otimes_B L_2) \xrightarrow{\ \sim\ } H^{i+j}_{m_C}(M \otimes_A (L_1 \otimes_B L_2)). \tag{22}$$

As in (21′), if $\mathbf{s}$ and $\mathbf{t}$ denote systems of parameters for m_B/m_AB and m_C/m_BC respectively, then (22) is described in terms of generalized fractions by the rule

$$\begin{bmatrix} \begin{bmatrix} m \otimes l_1 \\ \mathbf{s} \end{bmatrix} \otimes l_2 \\ \mathbf{t} \end{bmatrix} \longrightarrow \begin{bmatrix} m \otimes (l_1 \otimes l_2) \\ \mathbf{s}, \mathbf{t} \end{bmatrix}, \qquad \{m \in M, l_1 \in L_1, l_2 \in L_2\}. \tag{22$'$}$$

Finally, the isomorphism in (22) is compatible with passage to completion: Let $\widehat{A}, \widehat{B}, \widehat{C}$ denote the respective completions of A, B, C along the corresponding maximal ideals. Set $\widehat{L}_1 := L_1 \otimes_B \widehat{B}$, $\widehat{L}_2 := L_2 \otimes_C \widehat{C}$. The following diagram commutes

$$\begin{array}{ccc} H^j_{m_C}(H^i_{m_B}(M \otimes_A L_1) \otimes_B L_2) & \longrightarrow & H^{i+j}_{m_C}(M \otimes_A (L_1 \otimes_B L_2)) \\ \big\downarrow & & \big\downarrow \\ H^j_{m_{\widehat{C}}}(H^i_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{L}_1) \otimes_{\widehat{B}} \widehat{L}_2) & \longrightarrow & H^{i+j}_{m_{\widehat{C}}}(M \otimes_{\widehat{A}} (\widehat{L}_1 \otimes_{\widehat{B}} \widehat{L}_2)) \end{array} \tag{$\widehat{22}$}$$

where the rows are isomorphisms obtained using (22) and the columns are isomorphisms obtained by applying (19) iteratively.

For convenience of reference, we also recall the analogue of (22) in Huang's convention for local cohomology and generalized fractions. Thus in Huang's convention, where the order of the tensor product is reversed, the iterated isomorphism of (22) is written as (see [**13**, 2.5])

$$H^j_{m_C}(L_2 \otimes_B H^i_{m_B}(L_1 \otimes_A M)) \xrightarrow{\ \sim\ } H^{i+j}_{m_C}((L_2 \otimes_B L_1) \otimes_A M), \tag{22*}$$

and at the level of generalized fractions, is given by

$$\begin{bmatrix} l_2 \otimes \begin{bmatrix} l_1 \otimes m \\ \mathbf{s} \end{bmatrix} \\ \mathbf{t} \end{bmatrix} \longrightarrow \begin{bmatrix} (l_2 \otimes l_1) \otimes m) \\ \mathbf{t}, \mathbf{s} \end{bmatrix}, \qquad \{m \in M, l_1 \in L_1, l_2 \in L_2\}.$$

4.2.2. For A, B, C, M as in (22) there also exists a natural isomorphism

$$\mathrm{Hom}^{\mathrm{c}}_B(C, \mathrm{Hom}^{\mathrm{c}}_A(B, M)) \xrightarrow{\ \sim\ } \mathrm{Hom}^{\mathrm{c}}_A(C, M) \tag{23}$$

corresponding to "evaluation at 1". Furthermore the following diagram of isomorphisms induced by (20) and (23) commutes.

$$\begin{array}{ccc} \mathrm{Hom}^{\mathrm{c}}_B(C, \mathrm{Hom}^{\mathrm{c}}_A(B, M)) & \longrightarrow & \mathrm{Hom}^{\mathrm{c}}_A(C, M) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}^{\mathrm{c}}_{\widehat{B}}(\widehat{C}, \mathrm{Hom}^{\mathrm{c}}_{\widehat{A}}(\widehat{B}, M)) & \longrightarrow & \mathrm{Hom}^{\mathrm{c}}_{\widehat{A}}(\widehat{C}, M) \end{array} \tag{$\widehat{23}$}$$

4.2.3. Let A, B, M be as in (22), and let J be an ideal in A. Let L be a B-module. Set $\overline{A} := A/J$, $\overline{B} := B/JB$, $\overline{L} = L/JL$. Then for any integer r there is a natural map

$$H^r_{m_{\overline{B}}}(\mathrm{Hom}_A(\overline{A}, M) \otimes_{\overline{A}} \overline{L}) \longrightarrow \mathrm{Hom}_B(\overline{B}, H^r_{m_B}(M \otimes_A L)) \tag{24}$$

defined by the following sequence (where $N := \mathrm{Hom}_A(\overline{A}, M)$, $m = m_B$, $\overline{m} = m_{\overline{B}}$)

$$\begin{aligned} H^r_{\overline{m}}(N \otimes_{\overline{A}} \overline{L}) \cong H^r_m(N \otimes_{\overline{A}} \overline{L}) \cong H^r_m(N \otimes_A L) \cong \mathrm{Hom}_B(\overline{B}, H^r_m(N \otimes_A L)) \\ \longrightarrow \mathrm{Hom}_B(\overline{B}, H^r_m(M \otimes_A L)). \end{aligned}$$

(Note that for a surjective map $R \to \overline{R}$, continuous Hom = usual Hom, i.e., $\mathrm{Hom}^{\mathrm{c}}_R(\overline{R}, -) = \mathrm{Hom}_R(\overline{R}, -)$.) If we further assume that $A \to B$ is formally smooth and L is flat over B and if $r = \dim(B/m_A B)$ then (24) is an isomorphism ([**13**, 3.6]). Finally, the following diagram, induced by (19), (20) and (24) commutes.

$$\begin{array}{ccc} H^r_{m_{\overline{B}}}(\mathrm{Hom}_A(\overline{A}, M) \otimes_{\overline{A}} \overline{L}) & \longrightarrow & \mathrm{Hom}_B(\overline{B}, H^r_{m_B}(M \otimes_A L)) \\ \downarrow & & \downarrow \\ H^r_{m_{\overline{\widehat{B}}}}(\mathrm{Hom}_{\widehat{A}}(\overline{\widehat{A}}, M) \otimes_{\overline{\widehat{A}}} \overline{\widehat{L}}) & \longrightarrow & \mathrm{Hom}_{\widehat{B}}(\overline{\widehat{B}}, H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{L})) \end{array} \tag{$\widehat{24}$}$$

4.2.4. Let $A \xrightarrow{f} B \twoheadrightarrow A$ be local homomorphisms of noetherian local rings that factor the identity map on A. Suppose f is formally smooth of relative dimension r. Let $t_1, \ldots, t_r$ be a regular system of parameters of $m_B/m_A B$. Let M be an m_A-torsion A-module and L be a rank one free B-module with generator g. Then there exists a natural isomorphism (depending on the choices $t_1, \ldots, t_r$ and g)

$$\mathrm{Hom}_B(A, H^r_{m_B}(M \otimes_A L)) \xrightarrow{\ \sim\ } M \tag{25}$$

which we now describe in terms of generalized fractions. In this case it is convenient to first pass to the completions (see [**13**, Chp. 5] for proofs of statements below). Upon completing A, B, L we get that $\widehat{B}$ can be identified with a power series ring over $\widehat{A}$, say $\widehat{B} \cong \widehat{A}[[T_1, \ldots, T_r]]$ where t_i maps to T_i. Further, $H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{L})$, as an $\widehat{A}$-module, is isomorphic to a direct sum of copies of M, i.e., there is a natural decomposition

$$H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{L}) \cong \oplus M_{i_1,\ldots,i_r}\,, \qquad M_{i_1,\ldots,i_r} = \begin{cases} M, & \text{if } i_j > 0 \text{ for all } j; \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Moreover if $i_j > 0$ for all j, the canonical inclusion $M = M_{i_1,\ldots,i_r} \to H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{L})$ is given by

$$m \to \begin{bmatrix} m \otimes g \\ T_1^{i_1}, \ldots, T_r^{i_r} \end{bmatrix}, \qquad m \in M.$$

It follows that the map $M_{i_1,\ldots,i_r} \to M_{i_1,\ldots,i_j-1,\ldots,i_r}$, given by multiplication by T_j, is an isomorphism for $i_j > 1$ and zero for $i_j = 1$. In particular, the submodule of $H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{L})$ consisting of elements annihilated by $T_1, \ldots, T_r$ is precisely the summand $M_{1,\ldots,1}$. In view of (19) and (20), we obtain the isomorphism in (25). Now suppose $\widehat{L} = \widehat{\Omega}^r_{\widehat{B}/\widehat{A}}$ and $g = dT_1 \wedge \ldots \wedge dT_r$. Consider the natural maps

$$\mathrm{Hom}_{\widehat{B}}(\widehat{A}, H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{\Omega}^r_{\widehat{B}/\widehat{A}})) \xrightarrow{\ i\ } H^r_{m_{\widehat{B}}}(M \otimes_{\widehat{A}} \widehat{\Omega}^r_{\widehat{B}/\widehat{A}}) \xrightarrow{\ \mathrm{res}\ } M = M_{1,\ldots,1} \tag{$\widehat{25}$}$$

where i is the canonical inclusion and res is the projection map induced by (26).

Then res, which *a priori* depends on the choice of variables $T_1, \dots, T_r$, is in fact independent of such a choice[5]. Since res is canonical it follows that $\widehat{(25)}$ defines a canonical isomorphism. In terms of generalized fractions $\widehat{(25)}$ is given by

$$m \longrightarrow \begin{bmatrix} m \otimes dT_1 \wedge \ldots \wedge dT_r \\ T_1, \dots, T_r \end{bmatrix}. \tag{25$'$}$$

4.3. Pseudofunctors over local rings. We now describe a canonical pseudofunctor over local rings, one that forms the base for the pseudofunctor that we construct over the category $\mathbb{F}$ of formal schemes. Let $\mathfrak{C}$ denote the category whose objects are noetherian complete local rings and whose morphisms are local homomorphisms that are essentially of pseudo-finite type (see §2.1). By 2.4.5(i), our definition of $\mathfrak{C}$ agrees with the one in [**13**, p. 28]. Note that $\mathfrak{C}$ is anti-isomorphic to the full subcategory of connected zero-dimensional formal schemes in $\mathbb{F}$. By a smooth map in $\mathfrak{C}$ we mean a map which is formally smooth under the topology given by powers of the corresponding maximal ideals.

THEOREM 4.3.1. *For any ring $R \in \mathfrak{C}$, let $R_\sharp$ denote the category of zero-dimensional R-modules. Then there exists a canonical covariant pseudofunctor $(-)_\sharp$ on $\mathfrak{C}$ and a choice for isomorphisms as in* I(i) *and* I(ii) *below, such that* II-IV *below are satisfied. (The terms $C_\sharp^{(-),(-)}$ and $\delta_\sharp^{(-)}$ used in* II *refer to the comparison isomorphisms associated with $(-)_\sharp$).*

I. (i) *If $f\colon A \to B$ is a smooth map of relative dimension r in $\mathfrak{C}$, then for any $M \in A_\sharp$, there is a natural isomorphism (where $\omega_f = \widehat{\Omega}^r_{B/A}$)*

$$f_\sharp M \xrightarrow{\sim} H^r_{m_B}(M \otimes_A \omega_f).$$

(ii) *If $f\colon A \to B$ is a surjective map in $\mathfrak{C}$, then for any $M \in A_\sharp$, there is a natural isomorphism*

$$f_\sharp M \xrightarrow{\sim} \mathrm{Hom}_A(B, M).$$

II. (i) *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be smooth maps in $\mathfrak{C}$ having relative dimensions r_1, r_2 respectively. Let t_1, t_2 denote the transcendence degree of the induced maps of residue fields $k_A \to k_B$ and $k_B \to k_C$ respectively. Then for any $M \in A_\sharp$, the following diagram commutes*

$$\begin{tikzcd} g_\sharp f_\sharp M \arrow[r, "C_\sharp^{f,g}"] \arrow[d] & (gf)_\sharp M \arrow[d] \\ H^{r_2}_{m_C}(H^{r_1}_{m_B}(M \otimes_A \omega_f) \otimes_B \omega_g) \arrow[r] & H^{r_2+r_1}_{m_C}(M \otimes_A \omega_{gf}) \end{tikzcd}$$

where the vertical maps are obtained by using I.(i) *and the bottom row is $(-1)^{t_1 r_2}$ times the map given by* (22) *and the natural isomorphism $\omega_f \otimes_B \omega_g \xrightarrow{\sim} \omega_{gf}$ induced by the exact sequence in* 2.5.2.

[5]The map res is called the residue map.

(ii) *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be surjective homomorphisms in* $\mathfrak{C}$. *Then for any* $M \in A_\sharp$, *the following diagram, whose vertical maps are obtained using* I.(ii), *commutes.*

$$\begin{tikzcd} g_\sharp f_\sharp M \arrow[r, "C_\sharp^{f,g}"] \arrow[d] & (gf)_\sharp M \arrow[d] \\ \mathrm{Hom}_B(C, \mathrm{Hom}_A(B,M)) \arrow[r, "(23)"] & \mathrm{Hom}_A(C,M) \end{tikzcd}$$

(iii) *Let* f, A, B, M, r *be as in* I.(i). *Let* J *be an ideal in* A. *Let* $\overline{f}\colon \overline{A} \to \overline{B}$ *denote the map induced by going modulo* J. *Let* $A \xrightarrow{i} \overline{A}, B \xrightarrow{j} \overline{B}$ *denote the canonical surjections. Then for any* $M \in A_\sharp$, *the following diagram commutes*

$$\begin{tikzcd} \overline{f}_\sharp i_\sharp M \arrow[r, "(C_\sharp^{f,j})^{-1} C_\sharp^{i,\overline{f}}"] \arrow[d] & j_\sharp f_\sharp M \arrow[d] \\ H^r_{m_{\overline{B}}}(\mathrm{Hom}_A(\overline{A}, M) \otimes_{\overline{A}} \omega_{\overline{f}}) \arrow[r] & \mathrm{Hom}_B(\overline{B}, H^r_{m_B}(M \otimes_A \omega_f)) \end{tikzcd}$$

where the vertical maps are obtained using I.(i),(ii) *and the bottom row is given by* (24) *and the natural isomorphism* $\omega_f \otimes_B \overline{B} \xrightarrow{\sim} \omega_{\overline{f}}$.

(iv) *Let* $A \xrightarrow{f} B \xrightarrow{\pi} A$ *be maps in* $\mathfrak{C}$ *such that* $\pi f = 1_A$ *and* f *is smooth of relative dimension* r. *Then for any* $M \in A_\sharp$, *the following diagram, with vertical maps induced by* I.(i),(ii), *commutes.*

$$\begin{tikzcd} \pi_\sharp f_\sharp M \arrow[r, "C_\sharp^{f,\pi}"] \arrow[d] & M \arrow[d, "\delta_\sharp^A"] \\ \mathrm{Hom}_B(A, H^r_{m_B}(M \otimes_A \omega_f)) \arrow[r, "\widehat{(25)}"] & M \end{tikzcd}$$

III. *If* $f\colon A \to A$ *is the identity map, then for any* $M \in A_\sharp$, *the following isomorphisms*

$$f_\sharp M \xrightarrow{\text{by I,(i)}} H^0_{m_A} M \xrightarrow{\textit{canonical}} M, \qquad f_\sharp M \xrightarrow{\text{by I,(ii)}} \mathrm{Hom}_A(A, M) \xrightarrow{\textit{canonical}} M,$$

agree with the isomorphism induced by $\delta_\sharp^A\colon f_\sharp \xrightarrow{\sim} 1_{A_\sharp}$.

IV. *For any map* $f\colon A \to B$ *in* $\mathfrak{C}$, $f_\sharp$ *takes an injective hull of the residue field* k_A *of* A *to an injective hull of* k_B.

The proof of 4.3.1 is based on Huang's result [**13**, Theorem 6.12]. We also need the following lemma for the proof.

Lemma 4.3.2. *Let* $\left((-)_\alpha, (-)_\alpha, C_\alpha^{(-),(-)}, \delta_\alpha^{(-)}\right)$ *be a pseudofunctor on* $\mathfrak{C}$. *For any two maps* $R \xrightarrow{f} S \xrightarrow{g} T$ *set* $C_\beta^{f,g} := (-1)^{t_1 t_2} C_\alpha^{f,g}$ *where* t_1, t_2 *are the transcendence degrees of the induced maps of residue fields* $k_R \to k_S$ *and* $k_S \to k_T$ *respectively. Then* $\left((-)_\alpha, (-)_\alpha, C_\beta^{(-),(-)}, \delta_\alpha^{(-)}\right)$ *is also a pseudofunctor on* $\mathfrak{C}$.

Proof. This is a straightforward verification. □

Proof of 4.3.1. In [**13**], Huang constructs a canonical covariant pseudofunctor on $\mathfrak{C}$ which we denote, following Huang, by '$\#$'. For any ring R in $\mathfrak{C}$, $R_\#$ is the category of zero-dimensional R-modules and so $R_\# = R_\natural$. We define '$\natural$' as the pseudofunctor obtained by modifying '$\#$' using 4.3.2. Then for any map $f\colon A \to B$ in $\mathfrak{C}$, we have $f_\natural = f_\#$. In order to prove properties I–IV for '$\natural$' we now recall some properties of '$\#$'.

Let $f\colon R \to T$ be a morphism in $\mathfrak{C}$ and let $R \xrightarrow{\eta} S \overset{\pi}{\twoheadrightarrow} T$ be a factorization of f where η is a smooth map of relative dimension r. For any $M \in R_\#$, set

$$(\eta,\pi)_\# M := \operatorname{Hom}_S(T, H^r_{m_S}(\omega_\eta \otimes_R M))$$

(cf. [**13**, p. 29] where Huang uses the somewhat ambiguous notation $\eta^0_\#$ instead). The construction of $(-)_\#$ provides a natural isomorphism

$$f_\# M \xrightarrow{\sim} (\eta,\pi)_\# M, \qquad \text{see [\mathbf{13}, p. 33].} \tag{27}$$

Note that the order of the tensor product $\omega_\eta \otimes_R M$ occurring in $(\eta,\pi)_\# M$ is reverse to the one we work with, such as the one in I.(i) of 4.3.1. Our next task includes redescribing Huang's $(-)_\#$ in our convention.

For f as in I.(i), consider the factorization $A \xrightarrow{f} B \xrightarrow{=} B$. We define the required isomorphism in I.(i), to be given by the following sequence

$$f_\natural M = f_\# M \xrightarrow{(27)} (f, 1_B)_\# M \xrightarrow{\text{can.}} H^r_{m_B}(\omega_f \otimes_A M) \xrightarrow{\sim} H^r_{m_B}(M \otimes_A \omega_f) \tag{28}$$

where the last isomorphism is induced by the obvious map $\omega_f \otimes_A M \xrightarrow{\sim} M \otimes_A \omega_f$.

For f as in I.(ii), consider the factorization $A \xrightarrow{=} A \twoheadrightarrow B$. We define the required isomorphism in I.(ii), to be given by the following sequence

$$f_\natural M = f_\# M \xrightarrow{(27)} (1_A, f)_\# M \xrightarrow{\text{can.}} \operatorname{Hom}_A(B, M). \tag{29}$$

Now under the hypothesis of part II.(i) of the Theorem we claim that the following diagram commutes where the vertical maps are obtained by using (28) and the bottom row is $(-1)^{t_1(r_2+t_2)}$ times the map given by (22) and the canonical isomorphism $\omega_f \otimes_B \omega_g \xrightarrow{\sim} \omega_{gf}$ induced by the exact sequence in 2.5.2.

$$\begin{tikzcd}
g_\# f_\# M \arrow[r, "C^{f,g}_\#"] \arrow[d] & (gf)_\# M \arrow[d] \\
H^{r_2}_{m_C}(H^{r_1}_{m_B}(M \otimes_A \omega_f) \otimes_B \omega_g) \arrow[r] & H^{r_2+r_1}_{m_C}(M \otimes_A \omega_{gf})
\end{tikzcd} \tag{30}$$

Since $C^{f,g}_\natural = (-1)^{t_1 t_2} C^{f,g}_\#$, commutativity of (30) implies that of the diagram in II.(i). To prove that (30) commutes we shall expand it vertically using the definition of (28). First note that the following diagram commutes due to functorial reasons.

$$\begin{tikzcd}
g_\# f_\# M \arrow[r, "(28)"] \arrow[d, "(27)"', "g_\# f_\# \cong (g,1)_\#(f,1)_\#"] & g_\# H^{r_1}_{m_B}(M \otimes_A \omega_f) \arrow[d, "(28)"] \\
H^{r_2}_{m_C}(\omega_g \otimes_B H^{r_1}_{m_B}(\omega_f \otimes_A M)) \arrow[r, "\text{swap}"] & H^{r_2}_{m_C}(H^{r_1}_{m_B}(M \otimes_A \omega_f) \otimes_B \omega_g)
\end{tikzcd}$$

Therefore we may expand (30) as follows.

$$
\begin{array}{ccc}
g_\# f_\# M & \xrightarrow{C_\#^{f,g}} & (gf)_\# M \\
{\scriptstyle (27)}\big\downarrow {\scriptstyle g_\# f_\# \cong (g,1)_\# (f,1)_\#} & & {\scriptstyle (27)}\big\downarrow {\scriptstyle (gf)_\# \cong (gf,1)_\#} \\
H^{r_2}_{m_C}(\omega_g \otimes_B H^{r_1}_{m_B}(\omega_f \otimes_A M)) & \xrightarrow{\alpha} & H^{r_2+r_1}_{m_C}(\omega_{gf} \otimes_A M) \\
{\scriptstyle \beta}\big\downarrow & & {\scriptstyle \gamma}\big\downarrow \\
H^{r_2}_{m_C}(H^{r_1}_{m_B}(M \otimes_A \omega_f) \otimes_B \omega_g) & \xrightarrow{\delta} & H^{r_2+r_1}_{m_C}(M \otimes_A \omega_{gf})
\end{array}
\tag{31}
$$

The vertical maps are the obvious ones. Thus β, γ are induced by switching the order of the tensor products. The maps α, δ are defined as

$$
\begin{aligned}
\alpha &= (-1)^{r_1 t_2}\Big(\quad (22^*) \quad + \quad \omega_g \otimes_B \omega_f \xrightarrow{\text{canonical}} \omega_{gf}\Big), \\
\delta &= (-1)^{t_1(r_2+t_2)}\Big(\quad (22) \quad + \quad \omega_f \otimes_B \omega_g \xrightarrow{\text{canonical}} \omega_{gf}\Big).
\end{aligned}
$$

Then by the definition of $C_\#^{f,g}$ in [**13**, 6.10], the top rectangle of (31) commutes. For the bottom rectangle we use the following calculations on generalized fractions with notation explained below.

$$
\begin{aligned}
\begin{bmatrix} \mu \otimes \begin{bmatrix} \nu\otimes m \\ \mathbf{s} \end{bmatrix} \\ \mathbf{t} \end{bmatrix} &\xrightarrow{\alpha} (-1)^{r_1 t_2} \begin{bmatrix} (\mu \wedge \nu) \otimes m \\ \mathbf{t}, \mathbf{s} \end{bmatrix} \xrightarrow{\gamma} (-1)^{r_1 t_2} \begin{bmatrix} m \otimes (\mu \wedge \nu) \\ \mathbf{t}, \mathbf{s} \end{bmatrix} \\
&\qquad\qquad = (-1)^{r_1 r_2 + r_1 t_2} \begin{bmatrix} m \otimes (\mu \wedge \nu) \\ \mathbf{s}, \mathbf{t} \end{bmatrix}, \\
\begin{bmatrix} \mu \otimes \begin{bmatrix} \nu\otimes m \\ \mathbf{s} \end{bmatrix} \\ \mathbf{t} \end{bmatrix} &\xrightarrow{\beta} \begin{bmatrix} \begin{bmatrix} m\otimes\nu \\ \mathbf{s} \end{bmatrix} \otimes \mu \\ \mathbf{t} \end{bmatrix} \xrightarrow{\delta} (-1)^{t_1(r_2+t_2)} \begin{bmatrix} m \otimes (\nu \wedge \mu) \\ \mathbf{s}, \mathbf{t} \end{bmatrix} \\
&\qquad\qquad = (-1)^{r_1 r_2 + r_1 t_2} \begin{bmatrix} m \otimes (\mu \wedge \nu) \\ \mathbf{s}, \mathbf{t} \end{bmatrix}.
\end{aligned}
$$

(Here, $m \in M, \mu \in \omega_g, \nu \in \omega_f$ and $\mathbf{s}, \mathbf{t}$ are systems of parameters $B/m_A B, C/m_B C$ respectively and have lengths r_1, r_2 respectively. Also, the free modules $\widehat{\Omega}^1_{B/A}$ and $\widehat{\Omega}^1_{C/B}$ have ranks $r_1 + t_1, r_2 + t_2$ respectively.) Thus (30) commutes, thereby proving II.(i).

The remaining cases of part II of the Theorem are relatively straightforward. Note that for a closed immersion, the relative dimension is zero and so is the transcendence degree at the residue fields. Thus the sign change effected by 4.3.2 plays no role in the remaining cases of II. Similarly Huang's sign factor in [**13**, 6.10] does not affect the remaining cases. Thus (ii), (iii) and (iv) of II follow easily from Huang's definition of $C_\#^{f,g}$ (cf. 4.4, 3.6, 6.5 of [**13**]).

Part III immediately follows from the definition of $\delta_\sharp^A$ in [**13**, p. 37]. Part IV holds because $(-)_\#$ takes injective hulls to injective hulls. Finally, since $(-)_\#$ is canonical, so is $(-)_\sharp$. □

For convenience of reference we state the following easily verified remark.

Remark 4.3.3. Let $f\colon A \to B$ be an isomorphism in $\mathfrak{C}$. Then for $(-)_\sharp$ as in 4.3.1, for any $M \in A_\sharp$, the following two isomorphisms are equal.

$$
f_\sharp M \xrightarrow{\text{4.3.1, I.(i)}} H^0_{m_B}(M \otimes_A B) = M \otimes_A B \xrightarrow{\text{canonical}} M
$$

$$f_\sharp M \xrightarrow{4.3.1,\ \mathrm{I.(ii)}} \mathrm{Hom}_A(B, M) \xrightarrow{\text{canonical}} M$$

One proves 4.3.3 by verifying the analogous statement for Huang's pseudofunctor $(-)_\#$.

Remark 4.3.4. In I.(ii) of 4.3.1, since f is assumed to be a closed immersion, there is no distinction between the functors $\mathrm{Hom}_A(B, -)$ and $\mathrm{Hom}^{\mathrm{c}}_A(B, -)$. However, in general, the former does not preserve the property of modules being zero-dimensional and so the latter is preferred. Moreover we have the following.

Let $\mathfrak{C}_{\mathrm{rf}}$ be the subcategory of $\mathfrak{C}$ consisting of residually finite maps, i.e., those maps for which the corresponding induced map of residue fields is finite (e.g., a power series ring over the base ring). Then there is a canonical choice for a pseudofunctor on $\mathfrak{C}_{\mathrm{rf}}$, viz., one that assigns to any map $f\colon A \to B$ in $\mathfrak{C}_{\mathrm{rf}}$, the functor sending $M \in A_\#$ to $\mathrm{Hom}^{\mathrm{c}}_A(B, M) \in B_\#$, and for any pair of composable maps, the comparison map corresponding to "evaluation at 1" (see (23)). In [**13**, Chp. 7], Huang shows that the restriction of $(-)_\#$ to $\mathfrak{C}_{\mathrm{rf}}$ is in fact isomorphic to this canonical pseudofunctor. This isomorphism generalizes the one resulting from I.(ii) of 4.3.1.

4.4. Iterated fractions and translation invariance. The results in this subsection are somewhat technical in nature. The main result is 4.4.1.

Let $f\colon B \to C$ be a flat local homomorphism of noetherian local rings with corresponding maximal ideals m_B, m_C. Suppose $N^\bullet$ is a bounded-below complex of B-modules having the property that there exists an integer j such that $H^l_{m_B} N^\bullet = 0$ for $l \neq j$. This property is equivalent to requiring that there exist an integer j for which the natural truncation maps (§1.4, (x))

$$\mathbf{R}\varGamma_{m_B} N^\bullet \leftarrow \tau_{\le j} \mathbf{R}\varGamma_{m_B} N^\bullet \to (H^j_{m_B} N^\bullet)[-j] \tag{32}$$

are isomorphisms. This property holds for any shift of $N^\bullet$ too. Let $L^\bullet$ be a bounded-above complex of flat C-modules. From 3.1.6 we know that the natural map

$$\mathbf{R}\varGamma_{m_C}(\mathbf{R}\varGamma_{m_B} N^\bullet \otimes_B L^\bullet) \to \mathbf{R}\varGamma_{m_C}(N^\bullet \otimes_B L^\bullet)$$

is an isomorphism. Therefore, applying $H^i_{m_C}$ to the following natural map

$$(H^j_{m_B} N^\bullet)[-j] \otimes_B L^\bullet \xrightarrow{\cong} \mathbf{R}\varGamma_{m_B} N^\bullet \otimes_B L^\bullet \to N^\bullet \otimes_B L^\bullet \tag{33}$$

we obtain a natural isomorphism (for all i)

$$H^i_{m_C}((H^j_{m_B} N^\bullet)[-j] \otimes_B L^\bullet) \xrightarrow{\sim} H^i_{m_C}(N^\bullet \otimes_B L^\bullet). \tag{34}$$

We elaborate on this isomorphism in the special case when $N^\bullet$ and $L^\bullet$ consist of one module each. Let N be a B-module which, as a complex, satisfies the hypothesis of (32). For application purposes let q' denote the integer for which we have the isomorphism $\mathbf{R}\varGamma_{m_B} N \xrightarrow{\sim} (H^{q'}_{m_B} N)[-q']$ as in (32). Let a, b be integers. Let L be a flat C-module. For any integer p', we now consider a natural isomorphism

$$\theta_{a,b} : H^{p'}_{m_C}(H^{q'}_{m_B} N \otimes_B L) \xrightarrow{\sim} H^{p'+q'}_{m_C}(N \otimes_B L) \tag{35}$$

which is obtained by applying $H^{p'+q'-a-b}_{m_C}$ to the following sequence of maps described below. (We are also using the convention in §1.4, (viii).)

$$
\begin{aligned}
&\text{(35a)} \qquad (H^{q'}_{m_B}N \otimes_B L)[a-q'+b] \to (H^{q'}_{m_B}N)[a-q'] \otimes_B L[b] \\
&\text{(35b)} \qquad \to (H^{q'-a}_{m_B}(N[a]))[a-q'] \otimes_B L[b] \\
&\text{(35c)} \qquad \to N[a] \otimes_B L[b] \\
&\text{(35d)} \qquad \to (N \otimes_B L)[a+b]
\end{aligned}
$$

The maps (35a) and (35d) are the natural isomorphisms obtained using the conventions in §1.4, (v), while (35b) is an equality by §1.4, (viii). The map (35c) is obtained from (33) using $N^\bullet = N[a]$, $L^\bullet = L[b]$ and $j = q' - a$, and so from (34) we conclude that $H^{p'+q'-a-b}_{m_C}$ applied to (35c) yields an isomorphism. Thus (35) is an isomorphism.

Lemma 4.4.1. *In the above situation we have the following.*

(i) *The isomorphism $\theta_{a,b}$ in* (35) *is independent of the choice of a, b.*

(ii) *Assume the following.*

(a) *There exists an ideal $I \subset m_B$ such that N is I-torsion.*

(b) *There exists a sequence $\mathbf{s}$ of length q' in m_B such that $\mathbf{s}$ generates m_B/I up to radicals.*

(c) *There exists a sequence $\mathbf{t}$ of length p' in m_C such that $\mathbf{t}$ generates $m_C/(m_BC)$ up to radicals.*

Then $\theta_{a,b}$ satisfies the following iteration formula for generalized fractions

$$\theta(\begin{bmatrix} \begin{bmatrix} n \\ \mathbf{s} \end{bmatrix} \otimes l \\ \mathbf{t} \end{bmatrix}) \quad = \quad \begin{bmatrix} n \otimes l \\ \mathbf{s}, \mathbf{t} \end{bmatrix}, \qquad n \in N, \quad l \in L.$$

Proof. For part (i), it suffices to show that $\theta_{a,b} = \theta_{0,0}$. Let $\phi_{a,b}$ denote the map $(H^{q'}_{m_B}N \otimes_B L)[a-q'+b] \xrightarrow{\sim} (N \otimes_B L)[a+b]$ obtained by composing (35a)–(35d). To prove (i) it therefore suffices to show that $\phi_{a,b} = \phi_{0,0}[a+b]$ where the latter map denotes shifting $\phi_{0,0}$ by $a+b$. This reduces to verifying that the outer portion of the following diagram commutes

$$
\begin{tikzcd}
(H^{q'}_{m_B}N \otimes_B L)[a-q'+b] \arrow[r, "(35a)"] \arrow[d] & (H^{q'}_{m_B}N)[a-q'] \otimes_B L[b] \arrow[d, "(35b)"] \\
((H^{q'}_{m_B}N)[-q'] \otimes_B L)[a+b] \arrow[r] \arrow[d] & (H^{q'-a}_{m_B}(N[a]))[a-q'] \otimes_B L[b] \arrow[d, "(35c)"] \\
(N \otimes_B L)[a+b] & N[a] \otimes L[b] \arrow[l, "(35d)"']
\end{tikzcd}
$$

where the maps in left column describe $\phi_{0,0}[a+b]$, the remaining outer edges define $\phi_{a,b}$ and the horizontal map in the middle row is the composite of the following isomorphisms obtained using the conventions in §1.4, (iv), (viii).

$$
\begin{aligned}
((H^{q'}_{m_B}N)[-q'] \otimes_B L)[a+b] &= ((H^{q'-a}_{m_B}(N[a]))[-q'] \otimes_B L)[a+b] \\
&\xrightarrow{\sim} (H^{q'-a}_{m_B}(N[a]))[-q'][a] \otimes_B L[b]
\end{aligned}
$$

The upper rectangle commutes by Lemma 4.4.3 below, while the lower one commutes due to functoriality of the truncation maps. Thus (i) follows.

To prove (ii), by (i) we may assume $a = 0 = b$. By 4.1.1 we see that if the required iteration formula holds for K-fractions then it also holds for C-fractions. Therefore it suffices to prove (ii) using K-fractions.

Before we proceed further, we recall some terminology concerning the local cohomology functors. Let R be a commutative ring, $\mathbf{u}$ a finite sequence in R and let $\mathbf{u}R$ denote the ideal generated by $\mathbf{u}$. Then we set

$$H^i_{\mathbf{u}R}(-) := H^i\mathbf{R}\varGamma_{\mathbf{u}R}(-), \qquad H^i_{\underline{\mathbf{u}}}(-) := H^iK^\bullet_\infty(\mathbf{u}, -).$$

By 3.1.4 there is a natural isomorphism $H^i_{\mathbf{u}R}(-) \xrightarrow{\sim} H^i_{\underline{\mathbf{u}}}(-)$. If $R \to S$ is a homomorphism, then we distinguish the $K^\bullet_\infty$-complexes over R and S by using the terms $K^\bullet_\infty(\mathbf{u}; R)$ and $K^\bullet_\infty(\mathbf{u}; S)$ respectively, however for any S-module M, the term $H^i_{\underline{\mathbf{u}}}M$ is unambiguous in view of the canonical isomorphism

$$K^\bullet_\infty(\mathbf{u}; R) \otimes_R M \xrightarrow{\sim} K^\bullet_\infty(\mathbf{u}; S) \otimes_S M.$$

Returning to the proof of part (ii), we consider the following diagram through which we relate the map $\theta_{0,0}$ to a map involving the $K^\bullet_\infty$-complexes.

$$\begin{tikzcd}
H^{p'}_{m_C}(H^{q'}_{m_B}N \otimes_B L) \arrow[d, "\theta_{0,0}"] & \Big(K^\bullet_\infty(\mathbf{t}) \otimes_C ((K^\bullet_\infty(\mathbf{s}) \otimes_B N)[q'] \otimes_B L)\Big)[p'] \arrow[l, "\alpha"'] \arrow[d, "\gamma = ?"] \\
H^{p'+q'}_{m_C}(N \otimes_B L) & \Big(K^\bullet_\infty(\mathbf{t}, \mathbf{s}) \otimes_C (N \otimes_B L)\Big)[p' + q'] \arrow[l, "\delta"']
\end{tikzcd} \tag{36}$$

The objects in the left column are modules thought of as complexes existing in degree zero only. The objects in the right column are complexes of length $p'+q'$ since the sequences $\mathbf{t}$ and $\mathbf{s}$ have length p', q' respectively by assumption. Furthermore the complexes in the right column exist between degrees $-p' - q'$ and 0. The horizontal maps, α, β, described below in detail, are the natural truncation maps and are surjective at the graded level. Via the horizontal maps, the elements in the modules occurring in the left column can be represented by suitable Koszul fractions occurring in degree zero in the complexes in the right column. The proof of (ii) then reduces to first finding a map γ which makes (36) commute and then chasing the image of Koszul fractions through γ.

The horizontal maps in (36) are defined as follows. The map α is defined by the composition of the maps

$$\begin{aligned}
H^{p'}_{m_C}(H^{q'}_{m_B}N \otimes_B L) &\cong H^{p'}_{\mathbf{t}C}(H^{q'}_{\mathbf{s}B}N \otimes_B L) \\
&\cong H^{p'}_{\underline{\mathbf{t}}}(H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L) \\
&\leftarrow \Big(K^\bullet_\infty(\mathbf{t}) \otimes_C (H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L)\Big)[p'] \\
&\leftarrow \Big(K^\bullet_\infty(\mathbf{t}) \otimes_C ((K^\bullet_\infty(\mathbf{s}) \otimes_B N)[q'] \otimes_B L)\Big)[p']
\end{aligned}$$

where the last two maps are obtained by truncation of the appropriate $K^\bullet_\infty$-complex at the highest homology spot. The map δ in (36) is defined by the composition of the maps

$$\begin{aligned}
H^{p'+q'}_{m_C}(N \otimes_B L) \cong H^{p'+q'}_{(\mathbf{t},\mathbf{s})C}(N \otimes_B L) &\cong H^{p'+q'}_{\underline{\mathbf{t},\mathbf{s}}}(N \otimes_B L) \\
&\leftarrow \Big(K^\bullet_\infty(\mathbf{t}, \mathbf{s}) \otimes_C (N \otimes_B L)\Big)[p' + q'].
\end{aligned}$$

where the last map is obtained by truncating at degree zero.

In order to find γ which makes (36) commute we expand (36) horizontally using the definition of α and vertically using the definition of $\theta_{0,0}$. For convenience we do the expansion in two stages ((36a), (36b) below).

The leftmost column in (36a) corresponds to the definition of $\theta_{0,0}$. The horizontal maps on the left side in (36a) are, with the exception of the bottommost map δ_1, the isomorphisms induced by the natural maps $\mathbf{R}\varGamma_{m_C} \to \mathbf{R}\varGamma_{\mathbf{t}C}$ and $\mathbf{R}\varGamma_{m_B} \to \mathbf{R}\varGamma_{\mathbf{s}B}$ while δ_1 is induced by the natural map

$$\mathbf{R}\varGamma_{m_C} \to \mathbf{R}\varGamma_{(\mathbf{t},\mathbf{s})C} = \mathbf{R}(\varGamma_{\mathbf{t}C}\varGamma_{\mathbf{s}C}) \cong \mathbf{R}\varGamma_{\mathbf{t}C}\mathbf{R}\varGamma_{\mathbf{s}C}.$$

The vertical maps in the middle column, with the exception of the bottommost one γ_1, are defined similar to the way the corresponding ones on the left column are defined while γ_1 is induced by the natural isomorphism

$$\mathbf{R}\varGamma_{\mathbf{s}B}N \otimes_B L \cong \mathbf{R}\varGamma_{\mathbf{s}C}(N \otimes_B L).$$

The horizontal maps on the right side in (36a) are induced by the natural isomorphisms of the kind $\mathbf{R}\varGamma_?(-) \xrightarrow{\sim} K^\bullet_\infty(?) \otimes (-)$. The map γ_2 is induced by

$$\Theta\colon K^\bullet_\infty(\mathbf{t},(-))[-q'] \xrightarrow{\sim} K^\bullet_\infty(\mathbf{t},(-)[-q'])$$

where Θ is the second component of the δ-functor pair $(K^\bullet_\infty(\mathbf{t},(-)),\Theta)$, (cf. 3.1.4). The remaining vertical maps in the rightmost column of (36a) are the obvious counterparts of the maps in the middle column.

The top three rectangles on the left side and the bottom three rectangles on the right side in (36a) commute for obvious reasons. The topmost rectangle on the right commutes because the isomorphism in 3.1.4 is one of δ-functors. To verify the commutativity of the bottom rectangle on the left we expand it horizontally using obvious natural maps as follows.

$$\begin{array}{ccccc}
H^{p'+q'}_{m_C}(\mathbf{R}\varGamma_{m_B}N \otimes_B L) & \longrightarrow & H^{p'+q'}_{(\mathbf{t},\mathbf{s})C}(\mathbf{R}\varGamma_{\mathbf{s}B}N \otimes_B L) & \longrightarrow & H^{p'+q'}_{\mathbf{t}C}(\mathbf{R}\varGamma_{\mathbf{s}B}N \otimes_B L) \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\gamma_1} \\
H^{p'+q'}_{m_C}(N \otimes_B L) & \longrightarrow & H^{p'+q'}_{(\mathbf{t},\mathbf{s})C}(N \otimes_B L) & \longrightarrow & H^{p'+q'}_{\mathbf{t}C}\mathbf{R}\varGamma_{\mathbf{s}C}(N \otimes_B L)
\end{array}$$

In this diagram the rectangle on the left commutes for functorial reasons while the one on the right commutes by Lemma 4.4.2 below. Thus (36a) commutes.

For (36b) we use the notation

$$K^\bullet_1 := K^\bullet_\infty(\mathbf{t};C), \quad K^\bullet_2 := K^\bullet_\infty(\mathbf{s};C), \quad K^\bullet_N := K^\bullet_\infty(\mathbf{s};B) \otimes_B N.$$

The horizontal maps in (36b) are the obvious ones given by truncation. (In each case the complex in question has zero homology in higher degrees.) The leftmost column of (36b) is precisely the rightmost column of (36a). The remaining columns of (36b) are the obvious counterparts of its leftmost one. In particular, the convention in §1.4, (iv), applies for the maps $\gamma_3, \gamma_4, \gamma_5, \gamma_6$. Commutativity of (36b) is obvious from the functoriality of the maps involved.

Upon composing the top rows in (36a), (36b) we recover the map α of (36). In order to verify that the composition of the bottom rows in (36a), (36b) is the same as δ of (36) we reduce to checking commutativity of the following diagram whose left column gives δ and whose right column gives the sequence of bottom rows in

$$
\begin{tikzcd}
H^{p'}_{m_C}(H^{q'}_{m_B}N \otimes_B L) \arrow[r] \arrow[d, equal] & H^{p'}_{\mathbf{t}C}(H^{q'}_{\mathbf{s}B}N \otimes_B L) \arrow[r] \arrow[d, equal] & H^{p'}_{\underline{\mathbf{t}}}(H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L) \arrow[d, "\gamma_2"] \\
H^{p'+q'}_{m_C}((H^{q'}_{m_B}N \otimes_B L)[-q']) \arrow[r] \arrow[d, "(35a)"'] & H^{p'+q'}_{\mathbf{t}C}((H^{q'}_{\mathbf{s}B}N \otimes_B L)[-q']) \arrow[r] \arrow[d] & H^{p'+q'}_{\underline{\mathbf{t}}}((H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L)[-q']) \arrow[d] \\
H^{p'+q'}_{m_C}((H^{q'}_{m_B}N)[-q'] \otimes_B L) \arrow[r] & H^{p'+q'}_{\mathbf{t}C}((H^{q'}_{\mathbf{s}B}N)[-q'] \otimes_B L) \arrow[r] & H^{p'+q'}_{\underline{\mathbf{t}}}((H^{q'}_{\underline{\mathbf{s}}}N)[-q'] \otimes_B L) \\
H^{p'+q'}_{m_C}(\mathbf{R}\varGamma_{m_B}N \otimes_B L) \arrow[r] \arrow[u, "(32)"] \arrow[d, "\text{canonical}"'] & H^{p'+q'}_{\mathbf{t}C}(\mathbf{R}\varGamma_{\mathbf{s}B}N \otimes_B L) \arrow[r] \arrow[u, "\text{truncation}"'] \arrow[d, "\gamma_1"] & H^{p'+q'}_{\underline{\mathbf{t}}}((K^\bullet_\infty(\mathbf{s};B) \otimes_B N) \otimes_B L) \arrow[u, "\text{truncation}"'] \arrow[d, "\text{canonical}"] \\
H^{p'+q'}_{m_C}(N \otimes_B L) \arrow[r, "\delta_1"] & H^{p'+q'}_{\mathbf{t}C}\mathbf{R}\varGamma_{\mathbf{s}C}(N \otimes_B L) \arrow[r] & H^{p'+q'}_{\underline{\mathbf{t}}}(K^\bullet_\infty(\mathbf{s};C) \otimes_C (N \otimes_B L))
\end{tikzcd}
\tag{36a}
$$

$$
\begin{tikzcd}
H^{p'}_{\underline{\mathbf{t}}}(H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L) \arrow[d, "\gamma_2"'] & \Big(K^\bullet_1 \otimes_C (H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L)\Big)[p'] \arrow[l] \arrow[d, "\gamma_3"] & \Big(K^\bullet_1 \otimes_C (K^\bullet_N[q'] \otimes_B L)\Big)[p'] \arrow[l] \arrow[d, "\gamma_5"] \\
H^{p'+q'}_{\underline{\mathbf{t}}}((H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L)[-q']) \arrow[d] & \Big(K^\bullet_1 \otimes_C (H^{q'}_{\underline{\mathbf{s}}}N \otimes_B L)[-q']\Big)[p'+q'] \arrow[l] \arrow[d, "\gamma_4"] & \Big(K^\bullet_1 \otimes_C (K^\bullet_N[q'] \otimes_B L)[-q']\Big)[p'+q'] \arrow[l] \arrow[d, "\gamma_6"] \\
H^{p'+q'}_{\underline{\mathbf{t}}}((H^{q'}_{\underline{\mathbf{s}}}N)[-q'] \otimes_B L) & \Big(K^\bullet_1 \otimes_C ((H^{q'}_{\underline{\mathbf{s}}}N)[-q'] \otimes_B L)\Big)[p'+q'] \arrow[l] & \Big(K^\bullet_1 \otimes_C (K^\bullet_N \otimes_B L)\Big)[p'+q'] \arrow[l] \arrow[d, equal] \\
H^{p'+q'}_{\underline{\mathbf{t}}}(K^\bullet_N \otimes_B L) \arrow[u, "\text{truncation}"] \arrow[d, "\text{canonical}"'] & \Big(K^\bullet_1 \otimes_C (K^\bullet_N \otimes_B L)\Big)[p'+q'] \arrow[l] \arrow[u, "\text{truncation}"] \arrow[d, "\text{canonical}"'] \arrow[r, equal] & \Big(K^\bullet_1 \otimes_C (K^\bullet_N \otimes_B L)\Big)[p'+q'] \arrow[d, "\text{canonical}"] \\
H^{p'+q'}_{\underline{\mathbf{t}}}(K^\bullet_2 \otimes_C (N \otimes_B L)) & \Big(K^\bullet_1 \otimes_C (K^\bullet_2 \otimes_C (N \otimes_B L))\Big)[p'+q'] \arrow[l] & \Big(K^\bullet_\infty(\mathbf{t},\mathbf{s}) \otimes_C (N \otimes_B L)\Big)[p'+q'] \arrow[l, "\text{canonical}"']
\end{tikzcd}
\tag{36b}
$$

(36a)-(36d). (For simplicity, we set $n = p' + q'$, $M = N \otimes_B L$.)

$$
\begin{tikzcd}
H^n_{m_C} M \arrow[r, equal] \arrow[d] & H^n_{m_C} M \arrow[d] \\
H^n_{(\mathbf{t},\mathbf{s})C} M \arrow[r] \arrow[d] & H^n_{\mathbf{t}C} \mathbf{R}\Gamma_{\mathbf{s}C} M \arrow[d] \\
H^n_{\underline{\mathbf{t},\mathbf{s}}} M \arrow[r] & H^n_{\underline{\mathbf{t}}}(K^\bullet_\infty(\mathbf{s};C) \otimes_C M) \\
\Big(K^\bullet_\infty(\mathbf{t},\mathbf{s}) \otimes_C M\Big)[n] \arrow[r, equal] \arrow[u] & \Big(K^\bullet_\infty(\mathbf{t},\mathbf{s}) \otimes_C M\Big)[n] \arrow[u]
\end{tikzcd}
$$

The horizontal arrows in the middle are the obvious natural ones. Commutativity of the above diagram follows easily.

Since (36a) + (36b) = (36) holds along three edges of (36) we therefore have found a map γ which makes (36) commute, viz., γ is given by the rightmost column in (36b).

To complete the proof of (ii), let the sequence $\mathbf{s}$ be given by elements $s_1, \ldots, s_{q'}$ and $\mathbf{t}$ by $t_1, \ldots, t_{p'}$. Fix $n \in N, l \in L$. Set

$$
z_1 := \frac{\frac{n}{s_1 \cdots s_{q'}} \otimes l}{t_1 \cdots t_{p'}} \in F^0 \quad \text{where} \quad F^\bullet := \Big(K^\bullet_\infty(\mathbf{t}) \otimes_C ((K^\bullet_\infty(\mathbf{s}) \otimes_B N)[q'] \otimes_B L)\Big)[p']
$$

and

$$
z_2 := \frac{(-1)^{p'q'} n \otimes l}{t_1 \cdots t_{p'} \cdot s_1 \cdots s_{q'}} \in G^0 \quad \text{where} \quad G^\bullet := \Big(K^\bullet_\infty(\mathbf{t},\mathbf{s}) \otimes_C (N \otimes_B L)\Big)[p' + q'].
$$

The maps α and δ of (36) induce, in degree zero, the following maps of C-modules

$$
\alpha^0 \colon F^0 \to H^{p'}_{m_C}(H^{q'}_{m_B} N \otimes_B L), \quad \delta^0 \colon G^0 \to H^{p'+q'}_{m_C}(N \otimes_B L).
$$

From the definition of K-fractions in (16) we see that

$$
\alpha^0(z_1) = \begin{bmatrix} \left[\begin{smallmatrix} n \\ \mathbf{s} \end{smallmatrix}\right] \otimes l \\ \mathbf{t} \end{bmatrix} \quad \text{and} \quad \delta^0(z_2) = \begin{bmatrix} n \otimes l \\ \mathbf{s}, \mathbf{t} \end{bmatrix}.
$$

It therefore suffices to check that $\gamma^0(z_1) = z_2$ where γ^0 is the degree zero component of γ. Using that γ is given by the rightmost column in (36b), the verification $\gamma^0(z_1) = z_2$ follows easily. (Note that as per §1.4, (iv), the map $(\gamma_5)^0$ of (36b) has a sign of $(-1)^{p'q'}$ while $(\gamma_6)^0$ is the identity map.) $\square$

In the proof of 4.4.1, we used the following two lemmas.

LEMMA 4.4.2. *Let $B \to C$ be a homomorphism of noetherian rings. Let N be a B-module and let L be a flat C-module. For any ideals $\mathfrak{b} \subset B, \mathfrak{c} \subset C$, the following diagram, with maps described below, commutes.*

$$
\begin{tikzcd}
\mathbf{R}\Gamma_{\mathfrak{c}+\mathfrak{b}C}(\mathbf{R}\Gamma_{\mathfrak{b}} N \otimes_B L) \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & \mathbf{R}\Gamma_{\mathfrak{c}}(\mathbf{R}\Gamma_{\mathfrak{b}} N \otimes_B L) \arrow[d, "\alpha_3"'] \\
\mathbf{R}\Gamma_{\mathfrak{c}+\mathfrak{b}C}(N \otimes_B L) \arrow[r, "\alpha_4"] & \mathbf{R}\Gamma_{\mathfrak{c}}\, \mathbf{R}\Gamma_{\mathfrak{b}C}(N \otimes_B L)
\end{tikzcd}
$$

The map α_1 is induced by the natural map $\varGamma_{\mathfrak{c}+\mathfrak{b}C} \to \varGamma_{\mathfrak{c}}$. The map α_2 is induced by the natural map $\mathbf{R}\varGamma_{\mathfrak{b}} N \to N$. The map α_3 is induced by the natural isomorphism $\mathbf{R}\varGamma_{\mathfrak{b}} N \otimes_B L \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathfrak{b}C}(N \otimes_B L)$. The map α_4 is the natural isomorphism of derived functors corresponding to the composition $\varGamma_{\mathfrak{c}+\mathfrak{b}C} = \varGamma_{\mathfrak{c}}\,\varGamma_{\mathfrak{b}C}$. (The functor $\varGamma_{\mathfrak{b}C}$ sends injective modules to injective ones.)

Proof. Set $\mathfrak{b}' := \mathfrak{b}C$. We expand the above diagram vertically as follows where the unlabeled maps are the obvious natural ones.

$$
\begin{array}{ccc}
\mathbf{R}\varGamma_{\mathfrak{c}+\mathfrak{b}'}(\mathbf{R}\varGamma_{\mathfrak{b}} N \otimes_B L) & \xrightarrow{\alpha_1} & \mathbf{R}\varGamma_{\mathfrak{c}}\,(\mathbf{R}\varGamma_{\mathfrak{b}} N \otimes_B L) \\
\big\downarrow & & \big\downarrow{\scriptstyle\alpha_3} \\
\mathbf{R}\varGamma_{\mathfrak{c}+\mathfrak{b}'}\mathbf{R}\varGamma_{\mathfrak{b}'}(N \otimes_B L) & \longrightarrow & \mathbf{R}\varGamma_{\mathfrak{c}}\,\mathbf{R}\varGamma_{\mathfrak{b}'}(N \otimes_B L) \\
\big\downarrow & & \Big\| \\
\mathbf{R}\varGamma_{\mathfrak{c}+\mathfrak{b}'}(N \otimes_B L) & \xrightarrow{\alpha_4} & \mathbf{R}\varGamma_{\mathfrak{c}}\,\mathbf{R}\varGamma_{\mathfrak{b}'}(N \otimes_B L)
\end{array}
\tag{37}
$$

The top retangle obviously commutes. Set $M := N \otimes_B L$. Let $M \to I^\bullet$ be a C-injective resolution. Then $\mathbf{R}\varGamma_{\mathfrak{b}'}M = \varGamma_{\mathfrak{b}'}I^\bullet$. Since $\varGamma_{\mathfrak{b}'}I^\bullet$ also consists of C-injectives, the lower rectangle of (37) reduces to the following diagram, where each map being the obvious one, is an equality.

$$
\begin{array}{ccc}
\varGamma_{\mathfrak{c}+\mathfrak{b}'}\varGamma_{\mathfrak{b}'}I^\bullet & \longrightarrow & \varGamma_{\mathfrak{c}}\,\varGamma_{\mathfrak{b}'}I^\bullet \\
\big\downarrow & & \big\downarrow \\
\varGamma_{\mathfrak{c}+\mathfrak{b}'}I^\bullet & \longrightarrow & \varGamma_{\mathfrak{c}}\,\varGamma_{\mathfrak{b}'}I^\bullet
\end{array}
$$

□

The following lemma is proven by a direct calculation and we omit the proof.

Lemma 4.4.3. *Let R be a commutative ring. Let $F^\bullet$, $G^\bullet$ be complexes of R-modules. For any integers i, j, k the following diagram commutes*

$$
\begin{array}{ccc}
(F^\bullet \otimes G^\bullet)[i+j+k] & \xrightarrow{(\theta^{F,G}_{i+j,k})^{-1}} & F^\bullet[i+j] \otimes G^\bullet[k] \\
{\scriptstyle(\theta^{F,G}_{i,0})^{-1}[j+k]}\big\downarrow & & \Big\| \\
(F^\bullet[i] \otimes G^\bullet)[j+k] & \xrightarrow{(\theta^{F[i],G}_{j,k})^{-1}} & (F^\bullet[i])[j] \otimes G^\bullet[k]
\end{array}
$$

where all the maps are obtained using the convention in §1.4, (iv).

5. Pseudofunctorial behavior for smooth maps

The main result here is Proposition 5.2.1. In §5.1 we provide the main input that goes into defining the isomorphism (2) of §1.3 (see (38) below). In §5.2, we consider the situation of composition of two smooth maps. Suppose f, g are smooth maps such that gf exists. Then we consider the two Cousin-valued functors, one obtained by using 1.2.2(i) for gf and the other by using 1.2.2(i) iteratively for g and f. We provide a comparison isomorphism between these two in (39) below. In Proposition 5.2.1 we show that this global comparison map is compatible with the local one of (22) via the isomorphism in (38) below.

In particular, the results of this section allow us to define a Coz-valued pseudofunctor for the category of smooth maps in $\mathbb{F}_c$ (see §8.1). The proof of 5.2.1 is somewhat lengthy and §5.3 and §5.4 are devoted exclusively towards this. In §5.3 we decompose the diagram of 5.2.1 into convenient parts, the non-trivial cases being handled in §5.4.

5.1. An isomorphism. Let $h\colon (\mathcal{X}, \Delta_1) \to (\mathcal{Y}, \Delta)$ be a smooth map in $\mathbb{F}_c$ having constant relative dimension, say d. Let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}_\Delta(\mathcal{Y})$, i.e., $\mathcal{M}^\bullet$ is a Δ-Cousin complex of quasi-coherent torsion $\mathcal{O}_\mathcal{Y}$-modules. Let $\mathcal{L}$ be a flat quasi-coherent $\mathcal{O}_\mathcal{X}$-module. By 3.1.10, the complex $h^*\mathcal{M}^\bullet \otimes_\mathcal{X} \mathcal{L}[d]$ consists of $\mathcal{A}_{\mathrm{qc}}(\mathcal{X})$-modules and hence $E_{\Delta_1}\mathbf{R}\varGamma'_\mathcal{X}(h^*\mathcal{M}^\bullet \otimes_\mathcal{X} \mathcal{L}[d]) \in \mathrm{Coz}_{\Delta_1}(\mathcal{X})$. (see (12))

Let $x \in \mathcal{X}, y = h(x), p = \Delta_1(x), q = \Delta(y)$. We now define a natural isomorphism

$$(E_{\Delta_1}\mathbf{R}\varGamma'_\mathcal{X}(h^*\mathcal{M}^\bullet \otimes_\mathcal{X} \mathcal{L}[d]))(x) \xrightarrow{\sim} H^{p+d-q}_{m_x}(\mathcal{M}^\bullet(y) \otimes_y \mathcal{L}_x). \tag{38}$$

Set $M = \mathcal{M}^\bullet(y)$. Then (38) is the composition of the following isomorphisms (see explanatory remarks below).

$$
\begin{array}{llll}
 & (E_{\Delta_1}\mathbf{R}\varGamma'_\mathcal{X}(h^*\mathcal{M}^\bullet \otimes_\mathcal{X} \mathcal{L}[d]))(x) & & \\
\text{(38a)} & \xrightarrow{\sim} H^p_{m_x}(h^*\mathcal{M}^\bullet \otimes_\mathcal{X} \mathcal{L}[d])_x & & \text{(by (12))} \\
\text{(38b)} & \xrightarrow{\sim} H^p_{m_x}(h^*(i_yM)[-q] \otimes_\mathcal{X} \mathcal{L}[d])_x & & \text{(via truncation)} \\
\text{(38c)} & \xrightarrow{\sim} H^p_{m_x}(M[-q] \otimes_y \mathcal{L}_x[d]) & & \text{(stalk at } x) \\
\text{(38d)} & \xrightarrow{\sim} H^p_{m_x}((M \otimes_y \mathcal{L}_x)[d-q]) & & \text{(see §1.4, (v))} \\
\text{(38e)} & \xrightarrow{\sim} H^{p+d-q}_{m_x}(M \otimes_y \mathcal{L}_x). & & \text{(see §1.4, (viii))}
\end{array}
$$

For (38b) we use the sequence of natural maps

$$\mathcal{M}^\bullet \to \sigma_{\le q}\mathcal{M}^\bullet \leftarrow \sigma_{\ge q}\sigma_{\le q}\mathcal{M}^\bullet = \mathcal{M}^q[-q] \hookleftarrow i_yM[-q].$$

Each one of the complexes in this sequence is in $\mathrm{Coz}_\Delta(\mathcal{Y})$ and hence by (3.2.2), the functor $H^p_{m_x}(h^*(-) \otimes_\mathcal{X} \mathcal{L}[d])$ sends each map in this sequence to an isomorphism. Composing these isomorphisms appropriately results in (38b). The map in (38c) is induced by the canonical identification of the stalk at x (the stalk of i_yM at y being M). The map in (38d) is induced by the isomorphism (§1.4, (v))

$$M[-q] \otimes_y \mathcal{L}_x[d] \xrightarrow{\sim} (M \otimes_y \mathcal{L}_x)[d-q]$$

which, as per convention, is $(-1)^{qd}$ times the identity map on $M \otimes_y \mathcal{L}_x$.

Remark 5.1.1. Another way of obtaining the isomorphism from (38b)-(38c) is the following. Consider the complex $\mathcal{M}^\bullet_y$. Since $\mathcal{M}^\bullet$ is a Cousin complex, therefore $(\mathcal{M}^i)_y = 0$ for $i > q$ and furthermore $(\mathcal{M}^q)_y = \mathcal{M}^\bullet(y) = M$. In particular, there is

a natural map of complexes $M[-q] \to \mathcal{M}^\bullet_y$, and therefore an induced natural map $M[-q] \otimes_y \mathcal{L}_x[d] \to \mathcal{M}^\bullet_y \otimes_y \mathcal{L}_x[d]$. The following isomorphisms

$$
\begin{aligned}
&(38\mathrm{b}') \qquad H^p_{m_x}(h^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[d])_x \xrightarrow{\ \sim\ } H^p_{m_x}(\mathcal{M}^\bullet_y \otimes_y \mathcal{L}_x[d])\\
&(38\mathrm{c}') \qquad \xleftarrow{\ \sim\ } H^p_{m_x}(M[-q] \otimes_y \mathcal{L}_x[d])
\end{aligned}
$$

define the same map as the one resulting from (38b)–(38c).

Remark 5.1.2. By using (6) instead of (12) in (38a) we see that (38a)-(38e) in fact give an isomorphism

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(h^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[d]) \xrightarrow{\ \sim\ } H^{i+d-q}_{m_x}(\mathcal{M}^\bullet(y) \otimes_y \mathcal{L}_x)$$

for all i.

Lemma 5.1.3. *With notation as above, $\mathbf{R}\varGamma'_{\mathcal{X}}(h^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[d])$ is Cohen-Macaulay with respect to Δ_1 (§3.3). In other words, for any $x \in \mathcal{X}$ and any $i \in \mathbb{Z}$ such that $i \neq p = \Delta_1(x)$ we have*

$$H^i_x\mathbf{R}\varGamma'_{\mathcal{X}}(h^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[d]) \;=\; 0 \;=\; H^{i+d-q}_{m_x}(\mathcal{M}^\bullet(y) \otimes_y \mathcal{L}_x).$$

Proof. Set $M := \mathcal{M}^\bullet(y)$. We claim that for any finitely generated submodule N of M and any free $\mathcal{O}_{\mathcal{X},x}$-module L of finite rank, we have $H^{i+d-q}_{m_x}(N \otimes_y L) = 0$ for $i \neq p$. Since $M \otimes_y \mathcal{L}_x$ is a direct limit of modules of the type $N \otimes_y L$ and since local cohomology commutes with direct limits the lemma follows from the claim and 5.1.2.

Set $F := N \otimes_y L$. Since N has finite length it is a Cohen Macaulay $\mathcal{O}_{\mathcal{Y},y}$-module. By 2.4.5, (ii), the induced local homomorphism $\mathcal{O}_{\mathcal{Y},y} \to \mathcal{O}_{\mathcal{X},x}$ is flat. Moreover the fiber ring $\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x}$, being formally smooth over the residue field $\mathcal{O}_{\mathcal{Y},y}/m_y$, is a regular ring and hence a Cohen-Macaulay ring. Therefore, by [**9**, (6.3.3)] or [**19**, p. 181, Corollary], F is a Cohen Macaulay $\mathcal{O}_{\mathcal{X},x}$-module. In particular, $H^j_{m_x}F = 0$ for $j \neq \dim F$. It suffices to show that $\dim F := \dim \mathrm{Supp}(F) = p + d - q$.

Since N has finite length, we conclude that

$$\mathrm{Supp}(F) = \mathrm{Supp}((\mathcal{O}_{\mathcal{Y},y}/m_y) \otimes_y L) = \mathrm{Supp}(\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x}).$$

Therefore, by 2.6.10, $\dim(F) = p + d - q$. $\square$

5.2. Iteration of the Cousin functor. Let $(\mathcal{X}, \Delta_{\mathcal{X}}) \xrightarrow{f} (\mathcal{Y}, \Delta_{\mathcal{Y}}) \xrightarrow{g} (\mathcal{Z}, \Delta_{\mathcal{Z}})$ be smooth maps in $\mathbb{F}_c$ having constant relative dimension d, e respectively. Let $\mathcal{L}_1$ be a quasi-coherent flat $\mathcal{O}_{\mathcal{Y}}$-module and $\mathcal{L}_2$ a quasi-coherent flat $\mathcal{O}_{\mathcal{X}}$-module. For any complex $\mathcal{F}^\bullet$ of $\mathcal{O}_{\mathcal{Z}}$-modules and $\mathcal{G}^\bullet$ of $\mathcal{O}_{\mathcal{Y}}$-modules set

$$
\begin{aligned}
\mathbb{E}_g\mathcal{F}^\bullet &:= E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}(g^*\mathcal{F}^\bullet \otimes_{\mathcal{Y}} \mathcal{L}_1[e]),\\
\mathbb{E}_f\mathcal{G}^\bullet &:= E_{\Delta_{\mathcal{X}}}\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{G}^\bullet \otimes_{\mathcal{X}} \mathcal{L}_2[d]),\\
\mathbb{E}_{gf}\mathcal{F}^\bullet &:= E_{\Delta_{\mathcal{X}}}\mathbf{R}\varGamma'_{\mathcal{X}}((gf)^*\mathcal{F}^\bullet \otimes_{\mathcal{X}} (f^*\mathcal{L}_1 \otimes_{\mathcal{X}} \mathcal{L}_2)[d+e]).
\end{aligned}
$$

The functors $\mathbb{E}_{(-)}$ map Cousin complexes to Cousin complexes:

$$
\begin{aligned}
\mathbb{E}_g(\mathrm{Coz}_{\Delta_{\mathcal{Z}}}(\mathcal{Z})) &\subset \mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y}),\\
\mathbb{E}_f(\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})) &\subset \mathrm{Coz}_{\Delta_{\mathcal{X}}}(\mathcal{X}),\\
\mathbb{E}_{gf}(\mathrm{Coz}_{\Delta_{\mathcal{Z}}}(\mathcal{Z})) &\subset \mathrm{Coz}_{\Delta_{\mathcal{X}}}(\mathcal{X}).
\end{aligned}
$$

Let $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Z}}}(\mathcal{Z})$. There exists a natural isomorphism

$$\mathbb{E}_{gf}\mathcal{M}^\bullet \xrightarrow{\sim} \mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet \tag{39}$$

which we describe as follows : (with $\mathcal{G}^\bullet := g^*\mathcal{M}^\bullet \otimes_{\mathcal{Y}} \mathcal{L}_1[e]$)

$$\begin{aligned}
\mathbb{E}_{gf}\mathcal{M}^\bullet &:= E_{\Delta_{\mathcal{X}}}\mathbf{R}\Gamma'_{\mathcal{X}}((gf)^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} (f^*\mathcal{L}_1 \otimes_{\mathcal{X}} \mathcal{L}_2)[d+e]) \\
&\xrightarrow{\sim} E_{\Delta_{\mathcal{X}}}\mathbf{R}\Gamma'_{\mathcal{X}}(f^*g^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} (f^*\mathcal{L}_1[e] \otimes_{\mathcal{X}} \mathcal{L}_2[d])) && \text{(39a)}\\
&\xrightarrow{\sim} \mathbb{E}_f\mathcal{G}^\bullet && \text{(39b)}\\
&\xleftarrow{\sim} \mathbb{E}_f\mathbf{R}\Gamma'_{\mathcal{Y}}\mathcal{G}^\bullet && \text{(39c)}\\
&\xrightarrow{\sim} \mathbb{E}_f E_{\Delta_{\mathcal{Y}}}\mathbf{R}\Gamma'_{\mathcal{Y}}\mathcal{G}^\bullet =: \mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet && \text{(39d)}
\end{aligned}$$

The map (39a) is induced by $(f^*\mathcal{L}_1 \otimes_{\mathcal{X}} \mathcal{L}_2)[d+e] \xrightarrow{\sim} f^*\mathcal{L}_1[e] \otimes_{\mathcal{X}} \mathcal{L}_2[d]$ obtained from our convention in §1.4, (v). In particular, (39a) is $E\mathbf{R}\Gamma'_{\mathcal{X}}(-)$ of a map that is $(-1)^{ed}$ times the identity map at the graded level. The maps (39b), (39c) are the obvious natural ones, the latter being an isomorphism by 3.1.6. By 5.1.3, $\mathbf{R}\Gamma'_{\mathcal{Y}}\mathcal{G}^\bullet$ is a $\Delta_{\mathcal{Y}}$-Cohen-Macaulay complex on $\mathcal{Y}$. We define (39d) to be the natural map induced by the unique isomorphism $QE_{\Delta_{\mathcal{Y}}}\mathbf{R}\Gamma'_{\mathcal{Y}}\mathcal{G}^\bullet \cong \mathbf{R}\Gamma'_{\mathcal{Y}}\mathcal{G}^\bullet$ obtained from 3.3.2.

The isomorphism in (39) can be described explicitly at the punctual level. Fix a point x in $\mathcal{X}$ and let $y = f(x)$, $z = g(y)$. Let $\Delta_{\mathcal{Z}}(z) = r$, $\Delta_{\mathcal{Y}}(y) = q$, $\Delta_{\mathcal{X}}(x) = p$. Let $\mathcal{O}_{\mathcal{X},x}, \mathcal{O}_{\mathcal{Y},y}, \mathcal{O}_{\mathcal{Z},z}$ be the corresponding local rings and m_x, m_y, m_z the corresponding maximal ideals. Set $p' := p + d - q$ and $q' := q + e - r$.

Proposition 5.2.1. *The following diagram of $\mathcal{O}_{\mathcal{X},x}$-modules commutes.*

$$\begin{tikzcd}
(\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d, "(38)"] & & (\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) \arrow[ll, "(39)"'] \arrow[dd, "(38)"] \\
H^{p'}_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)(y) \otimes_y \mathcal{L}_{2x}) \arrow[d, "induced\ by\ (38)"] & & \\
H^{p'}_{m_x}(H^{q'}_{m_y}(\mathcal{M}^\bullet(z) \otimes_z \mathcal{L}_{1y}) \otimes_y \mathcal{L}_{2x}) \arrow[rr, "(22)"] & & H^{p'+q'}_{m_x}(\mathcal{M}^\bullet(z) \otimes_z (\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x}))
\end{tikzcd}$$

The rest of this section is devoted to proving Proposition 5.2.1. In 5.3 we indicate how to decompose the diagram in Proposition 5.2.1 into more convenient parts whose commutativity is verified in the last subsection.

5.3. Scheme of proof of 5.2.1. We now outline the general scheme of our proof of Proposition 5.2.1. The diagram, whose commutativity is in question, can be written as follows.

$$\begin{tikzcd}
V_1 \arrow[d] & V_2 \arrow[l] \arrow[dd] \\
V_3 \arrow[d] & \\
V_4 \arrow[r] & V_5
\end{tikzcd} \tag{40}$$

Without loss of generality, we assume that $\mathcal{M}^\bullet = i_z M[-r]$ where $M = \mathcal{M}^\bullet(z)$. This can be justified using the truncation arguments of (38b) as each truncation

map is functorial and the corresponding induced map on the cohomologies is an isomorphism. In order to prove commutativity of (40), we expand it in the following way explained below.

$$
\begin{tikzcd}
V_1 \arrow[r, equal] \arrow[d] & V_1 \arrow[d] & V_2 \arrow[l] \arrow[r, equal] \arrow[d] & V_2 \arrow[dd] \\
V_3 \arrow[d] & V_6 \arrow[r] \arrow[d] & V_7 \arrow[d] & \\
V_4 \arrow[r, equal] & V_4 \arrow[r] & V_8 \arrow[r] & V_5
\end{tikzcd}
\tag{41}
$$

For the above diagrams and subsequent ones occurring in this section, we use the notation that any map $V_i \to V_j$ occurring in the diagrams is denoted by $e(i,j)$. Set $M := \mathcal{M}^\bullet(z)$ and set $N := M \otimes_z \mathcal{L}_{1y}$. The vertices occurring in (41) are the following.

$$
\begin{aligned}
&V_1 := (\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) && V_5 := H^{p'+q'}_{m_x}(M \otimes_z (\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x}))\\
&V_2 := (\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) && V_6 := H^{p}_{m_x}((H^q_{m_y}(N[e-r]))[-q] \otimes_y \mathcal{L}_{2x}[d])\\
&V_3 := H^{p'}_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)(y) \otimes_y \mathcal{L}_{2x}) && V_7 := H^{p}_{m_x}(N[e-r] \otimes_y \mathcal{L}_{2x}[d])\\
&V_4 := H^{p'}_{m_x}(H^{q'}_{m_y}N \otimes_y \mathcal{L}_{2x}) && V_8 := H^{p'+q'}_{m_x}(N \otimes_y \mathcal{L}_{2x})
\end{aligned}
$$

The maps in (41) which come from (40), viz., $e(2,1), e(1,3), e(3,4)$ and $e(2,5)$ are, by definition, the ones specified in 5.2.1. The remaining maps in (41) are defined as follows.

The map $e(1,6)$ is defined as the composition of the following maps

$$
\begin{aligned}
V_1 = (\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) &\xrightarrow{\sim} H^p_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)(y)[-q] \otimes_y \mathcal{L}_{2x}[d])\\
&\xrightarrow{\sim} H^p_{m_x}((H^q_{m_y}(N[e-r]))[-q] \otimes_y \mathcal{L}_{2x}[d]) = V_6
\end{aligned}
$$

where the first isomorphism is obtained from the sequence (38a) to (38c) for the functor $\mathbb{E}_f$ acting on the Cousin complex $\mathbb{E}_g\mathcal{M}^\bullet$ on $\mathscr{Y}$, while the second isomorphism is induced from the sequence (38a) to (38d) for the functor $\mathbb{E}_g$ acting on $\mathcal{M}^\bullet$.

The map $e(2,7)$ is defined to be the composition of

$$
\begin{aligned}
V_2 = (\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) &\xrightarrow{\sim} H^p_{m_x}(M[-r] \otimes_z (\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x})[d+e])\\
&\xrightarrow{\sim} H^p_{m_x}(M[-r] \otimes_z (\mathcal{L}_{1y}[e] \otimes_y \mathcal{L}_{2x}[d]))\\
&\xrightarrow{\sim} H^p_{m_x}((M \otimes \mathcal{L}_{1y})[e-r] \otimes_y \mathcal{L}_{2x}[d]) = V_7
\end{aligned}
$$

where the first isomorphism is obtained using the sequence (38a) to (38c) while the second and the third maps are induced naturally by the isomorphisms

$$
(\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x})[d+e] \xrightarrow{\sim} \mathcal{L}_{1y}[e] \otimes_y \mathcal{L}_{2x}[d], \qquad M[-r] \otimes_z \mathcal{L}_{1y}[e] \xrightarrow{\sim} (M \otimes \mathcal{L}_{1y})[e-r]
$$

obtained using §1.4, (v).

We define $e(6,7)$ using (34) with the following notation. Let $B \to C$ denote the natural local homomorphism $\mathcal{O}_{\mathscr{Y},y} \to \mathcal{O}_{\mathscr{X},x}$. Let $N^\bullet = N[e-r]$, $L^\bullet = \mathcal{L}_{2x}[d]$ and $j = q$. The hypothesis of (34), which in effect is the hypothesis of (32), is satisfied because for any integer l, there are isomorphisms

$$
H^l_{m_B}N^\bullet := H^l_{m_y}(N[e-r]) = H^{l+e-r}_{m_y}(M \otimes_z \mathcal{L}_{1y}) \xrightarrow[5.1.2]{\sim} H^l_y\mathbf{R}\varGamma'_{\mathscr{Y}}(g^*\mathcal{M}^\bullet \otimes_{\mathscr{Y}} \mathcal{L}_1[e]),
$$

so that by 5.1.3, these modules vanish for $l \neq q$.

The map $e(6,4)$ is obtained as the composition of the maps

$$\begin{aligned} V_6 = H^p_{m_x}\Big((H^q_{m_y}(N[e-r]))[-q] \otimes_y \mathcal{L}_{2x}[d]\Big) &\xrightarrow{\ \sim\ } H^p_{m_x}((H^{q'}_{m_y}N)[-q] \otimes_y \mathcal{L}_{2x}[d]) \\ &\xrightarrow{\ \sim\ } H^{p'}_{m_x}(H^{q'}_{m_y}N \otimes_y \mathcal{L}_{2x}) = V_4 \end{aligned}$$

where the first isomorphism involves §1.4, (viii), as in (38e), while the second one is an imitation of (38d)-(38e).

For $e(7,8)$ we follow (38d)-(38e), while $e(4,8)$ is gotten using (21). The canonical associativity map for tensor products gives $e(8,5)$.

Note that $e(8,5)e(4,8) = e(4,5)$, where the latter is an arrow in (40). Therefore, to prove that (40) commutes it suffices to show that the four subdiagrams in (41) commute. Of these four, the ones on the extreme left and extreme right may be written as follows.

$$\text{(42)}\qquad \begin{array}{ccc} V_3 & \longleftarrow & V_1 \\ \downarrow & & \downarrow \\ V_4 & \longleftarrow & V_6 \end{array} \qquad\qquad \begin{array}{ccc} V_2 & \longrightarrow & V_5 \\ \downarrow & & \uparrow \\ V_7 & \longrightarrow & V_8 \end{array}$$

Commutativity of these two is easily verified; we do so in this subsection. Commutativity of the remaining two subdiagrams of (41) is proved in the next subsection.

The commutativity of the rectangle on the left in (42) follows from the commutativity of the following diagram whose maps are obtained using the various isomorphisms (38a) to (38e).

$$\begin{tikzcd} V_1 \arrow[d, "\text{(38a)-(38e)}"'] \arrow[rr, equal] & & V_1 \arrow[d, "\text{(38a)-(38c)}"'] \\ V_3 \arrow[d, "\text{(38a)-(38d)}"'] & & H^p_{m_x}\Big((\mathbb{E}_g\mathcal{M}^\bullet)(y)[-q] \otimes_y \mathcal{L}_{2x}[d]\Big) \arrow[ll, "\text{(38d)-(38e)}"'] \arrow[d, "\text{(38a)-(38d)}"'] \\ H^{p'}_{m_x}\Big((H^q_{m_y}(N[e-r])) \otimes_y \mathcal{L}_{2x}\Big) \arrow[d, "\text{(38e)}"'] & & V_6 \arrow[ll, "\text{(38d)-(38e)}"'] \arrow[d, "\text{(38e)}"'] \\ V_4 & & H^p_{m_x}\Big((H^{q'}_{m_y}N)[-q] \otimes_y \mathcal{L}_{2x}[d]\Big) \arrow[ll, "\text{(38d)-(38e)}"'] \end{tikzcd}$$

For the rectangle on the right in (42), we expand vertically using the definition of the vertical maps $e(2,7)$ and $e(8,5)$. Since both the maps $e(2,7)$ and $e(2,5)$ start from V_2 and factor through the steps (38a)-(38c), we may replace V_2 by the target of (38c). The resulting diagram, shown below, is seen to be commutative from the sign chase indicated along the inner sides of the arrows.

$$\begin{tikzcd} H^p_{m_x}(M[-r] \otimes_z (\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x})[d+e]) \arrow[r, "\text{(38d)-(38e)}", "(-1)^{(d+e)r}"'] \arrow[d, "(-1)^{de}"] & H^{p'+q'}_{m_x}(M \otimes_z (\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x})) \\ H^p_{m_x}(M[-r] \otimes_z (\mathcal{L}_{1y}[e] \otimes_y \mathcal{L}_{2x}[d])) \arrow[d, "(-1)^{re}"] & H^{p'+q'}_{m_x}(N \otimes_y \mathcal{L}_{2x}) \arrow[u] \\ H^p_{m_x}(N[e-r] \otimes_y \mathcal{L}_{2x}[d]) \arrow[r, "(-1)^{(e-r)d}", "\text{(38d)-(38e)}"'] & H^{p'+q'}_{m_x}(N \otimes_y \mathcal{L}_{2x}) \arrow[u, equal] \end{tikzcd}$$

5.4. Conclusion of proof of 5.2.1. In the previous subsection, the commutativity of the two rectangles in (42) was shown. Now we verify the commutativity of the remaining two subdiagrams in (41).

First we consider the following rectangle in (41).

$$
\begin{tikzcd}
V_6 \arrow[r] \arrow[d] & V_7 \arrow[d] \\
V_4 \arrow[r] & V_8
\end{tikzcd}
\tag{43}
$$

We claim that the isomorphism $e(7,8)e(6,7)e(6,4)^{-1}\colon V_4 \to V_8$ is the same as the isomorphism $\theta_{a,b}$ of (35) for the choices $a := e-r$, $b := d$ along with the notation $p' = p+d-q$, $q' = q+e-r$. Indeed, from the definition of the maps involved we see that $e(6,4)^{-1}$ corresponds to (35a)-(35b), $e(6,7)$ corresponds to (35c) and $e(7,8)$ corresponds to (35d). Now the commutativity of (43) follows from part (ii) of Lemma 4.4.1.

Thus the only remaining rectangle in (41) whose commutativity needs to be checked is the following.

$$
\begin{tikzcd}
V_1 \arrow[d] & V_2 \arrow[l] \arrow[d] \\
V_6 \arrow[r] & V_7
\end{tikzcd}
\tag{44}
$$

Before proceeding further we review our terminology. Recall that $\mathcal{M}^\bullet$ is assumed to be based at one point only, i.e., $\mathcal{M}^\bullet = i_z M[-r]$ for a suitable $\mathcal{O}_{\mathcal{Z},z}$-module M. Also $N = M \otimes_z \mathcal{L}_{1y}$. Set $\mathcal{N}^\bullet := (g^*\mathcal{M} \otimes_{\mathcal{Y}} \mathcal{L}_1)[e-r]$ where $\mathcal{M} = i_z M$. We shall henceforth make the canonical identification

$$
\mathcal{N}^\bullet_y = N[e-r]. \tag{45}
$$

Also note that with $\mathcal{G}^\bullet := g^*\mathcal{M}^\bullet \otimes_{\mathcal{Y}} \mathcal{L}_1[e]$ as in (39), using (§1.4, (v)), we obtain an isomorphism

$$
\mathcal{G}^\bullet := g^*\mathcal{M}^\bullet \otimes_{\mathcal{Y}} \mathcal{L}_1[e] \xrightarrow{\ \sim\ } (g^*\mathcal{M} \otimes_{\mathcal{Y}} \mathcal{L}_1)[e-r] =: \mathcal{N}^\bullet. \tag{46}
$$

Now expand (44) modulo (45) as follows, with the maps α_i described below.

$$
\begin{tikzcd}
(\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[dd, "\alpha_1"'] & (\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) \arrow[l, "e(2,1)"'] \\
 & H^p_{m_x}(\mathcal{N}^\bullet_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[u, "\alpha_2 = e(2,7)^{-1}"'] \\
H^p_{m_x}((E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y \otimes_y \mathcal{L}_{2x}[d]) & H^p_{m_x}((\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[l, "\alpha_4"'] \arrow[u, "\alpha_3"'] \\
H^p_{m_x}((H^q_{m_y}\mathcal{N}^\bullet_y)[-q] \otimes_y \mathcal{L}_{2x}[d]) \arrow[u, "\alpha_5"] & H^p_{m_x}(\mathbf{R}\varGamma_{m_y}\mathcal{N}^\bullet_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[l, "\alpha_7"'] \arrow[u, "\alpha_6"']
\end{tikzcd}
\tag{47}
$$

The map α_1 is the composition of the following isomorphisms

$$\begin{aligned}(\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) &\xrightarrow{(38a)} H^p_{m_x}(f^*\mathbb{E}_g\mathcal{M}^\bullet\otimes_{\mathcal{X}}\mathcal{L}_2[d])_x\\ &\xrightarrow[5.1.1]{(38b')} H^p_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)_y\otimes_y \mathcal{L}_{2x}[d])\\ &\longrightarrow H^p_{m_x}((E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y\otimes_y\mathcal{L}_{2x}[d])\end{aligned}$$

where the last map is the obvious one induced by (46). The map α_2 is the inverse of the isomorphism $e(2,7)$ via (45) while α_3 is the obvious natural map. The map α_4 is obtained by applying the isomorphism of functors $QE_{\Delta_{\mathcal{Y}}}\cong 1_{\mathbf{D}^+(\mathcal{Y})_{\mathrm{CM}}}$ of 3.3.2 on the Cohen-Macaulay complex $\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet$. The map α_5 is induced by the composition

$$(H^q_{m_y}\mathcal{N}^\bullet_y)[-q] \xrightarrow[(12)]{\sim} (E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)(y)[-q] \xrightarrow[\text{cf. 5.1.1}]{\text{truncation}} (E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y.$$

The map α_6 is the composition

$$\mathbf{R}\varGamma_{m_y}\mathcal{N}^\bullet_y \xrightarrow{3.1.5(iii)} (\mathbf{R}\varGamma_{\mathcal{J}}\,\mathcal{N}^\bullet)_y \xrightarrow{\text{canonical}} (\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y$$

where $\mathcal{J}$ is an open coherent ideal in $\mathcal{O}_{\mathcal{Y}}$ defining $\overline{\{y\}}$. Finally, α_7 is the isomorphism induced by the truncation maps in (32).

Now we verify that the outer border of (47) is equivalent to (44), which is the same as verifying the relations, $\alpha_5^{-1}\alpha_1 = e(1,6)$ and $\alpha_3\alpha_6\alpha_7^{-1} = e(6,7)$. The latter follows from the definitions involved while the former is a consequence of the commutativity of the following diagram. Here the left column represents $\alpha_5^{-1}\alpha_1$ while the right column represents $e(1,6)$ modulo (45). Also, recall that $\mathcal{M}^\bullet$ is assumed to be based only at z.

$$\begin{tikzcd}
(\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[r, equal] \arrow[d, "\text{(38a) and (38b}')"'] & (\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d, "\text{(38a)-(38c)}"'] \\
H^p_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)_y\otimes_y\mathcal{L}_{2x}[d]) \arrow[d, "\text{induced by (46)}"'] & H^p_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)(y)[-q]\otimes_y\mathcal{L}_{2x}[d]) \arrow[l, "(38c')^{-1}"'] \arrow[d, "\text{induced by (46)}"'] \\
H^p_{m_x}((E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y\otimes_y\mathcal{L}_{2x}[d]) & H^p_{m_x}((E_{\Delta_{\mathcal{Y}}}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)(y)[-q]\otimes_y\mathcal{L}_{2x}[d]) \arrow[l, "\text{via}"', "\text{truncation}"] \\
H^p_{m_x}((H^q_{m_y}\mathcal{N}^\bullet_y)[-q]\otimes_y\mathcal{L}_{2x}[d]) \arrow[u, "\alpha_5"] \arrow[r, equal] & H^p_{m_x}((H^q_{m_y}\mathcal{N}^\bullet_y)[-q]\otimes_y\mathcal{L}_{2x}[d]) \arrow[u, "\text{induced by (12)}", "\text{cf. (38a)}"']
\end{tikzcd}$$

To prove commutativity of the upper subdiagram in (47) we expand it horizontally using the definition of $e(2,1)$ and vertically using the definition of α_1. For convenience we break the diagram into two parts, see (48), (49) below.

The leftmost column of vertical maps in (48) defines α_1 of (47). The remaining columns follow the same pattern of definition. The horizontal maps on the left are induced by the isomorphism $QE_{\Delta_{\mathcal{X}}}\cong 1$ of 3.3.2 applied to $\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{G}^\bullet, \mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet$, while the ones on the right are the canonical ones induced by $\mathbf{R}\varGamma'_{\mathcal{Y}}\to 1$. In particular the horizontal maps in the bottommost row are α_3, α_4 of (47). It is obvious that (48) commutes.

The rightmost column of (48) is the same as the leftmost column of (49). We use $E' := E_{\Delta_{\mathcal{X}}}\mathbf{R}\varGamma'_{\mathcal{X}}$ for convenience. The remaining maps in (49) are the obvious ones as indicated by the labels. (The minus signs refer to the convention in §1.4(v).) Commutativity of (49) follows easily. Moreover, traveling along the bottom row

$$
\begin{tikzcd}
(\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d, "(38a)"', "\text{see (12)}"] & (\mathbb{E}_f\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{G}^\bullet)(x) \arrow[l, "(39d)"'] \arrow[r, "(39c)"] \arrow[d, "(12)"] & (\mathbb{E}_f\mathcal{G}^\bullet)(x) \arrow[d, "(12)"] \\
H^p_{m_x}(f^*\mathbb{E}_g\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}_2[d])_x \arrow[d, "(38b')"', "= \text{localization}"] & H^p_{m_x}(f^*\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{G}^\bullet \otimes_{\mathcal{X}} \mathcal{L}_2[d])_x \arrow[l] \arrow[r] \arrow[d, "\text{localization}"] & H^p_{m_x}(f^*\mathcal{G}^\bullet \otimes_{\mathcal{X}} \mathcal{L}_2[d])_x \arrow[d, "\text{localization}"] \\
H^p_{m_x}((\mathbb{E}_g\mathcal{M}^\bullet)_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[d, "\text{via (46)}"] & H^p_{m_x}((\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{G}^\bullet)_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[l] \arrow[r] \arrow[d, "\text{via (46)}"] & H^p_{m_x}(\mathcal{G}^\bullet_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[d, "\text{via (46)}"] \\
H^p_{m_x}((E_{\Delta_y}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y \otimes_y \mathcal{L}_{2x}[d]) & H^p_{m_x}((\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{N}^\bullet)_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[l, "\alpha_4"'] \arrow[r, "\alpha_3"] & H^p_{m_x}(\mathcal{N}^\bullet_y \otimes_y \mathcal{L}_{2x}[d])
\end{tikzcd}
\tag{48}
$$

$$
\begin{tikzcd}
(\mathbb{E}_f\mathcal{G}^\bullet)(x) \arrow[r, "(39b)"] \arrow[d, "(12)"] & (E'(f^*g^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} (f^*\mathcal{L}_1[e] \otimes_{\mathcal{X}} \mathcal{L}_2[d])))(x) \arrow[r, "(39a)"] \arrow[d, "(12)"] & (\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) \arrow[d, "\text{see (38a)}"', "(12)"] \\
H^p_{m_x}(f^*\mathcal{G}^\bullet \otimes_{\mathcal{X}} \mathcal{L}_2[d])_x \arrow[r] \arrow[d, "\text{localization}"] & H^p_{m_x}(f^*g^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} (f^*\mathcal{L}_1[e] \otimes_{\mathcal{X}} \mathcal{L}_2[d]))_x \arrow[r, "(-1)^{de}"'] \arrow[d, "\text{localization}"] & H^p_{m_x}((gf)^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} (f^*\mathcal{L}_1 \otimes_{\mathcal{X}} \mathcal{L}_2)[d+e])_x \arrow[d, "\text{localization}"] \\
H^p_{m_x}(\mathcal{G}^\bullet_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[d, "\text{via (46)}"', "(-1)^{re}"] & H^p_{m_x}((g^*\mathcal{M}^\bullet)_y \otimes_y (\mathcal{L}_{1y}[e] \otimes_y \mathcal{L}_{2x}[d])) \arrow[d, "(-1)^{re}"] & H^p_{m_x}(M[-r] \otimes_z (\mathcal{L}_{1y} \otimes_y \mathcal{L}_{2x})[d+e]) \arrow[d, "(-1)^{de}"] \\
H^p_{m_x}(\mathcal{N}^\bullet_y \otimes_y \mathcal{L}_{2x}[d]) \arrow[r, "(45)"] & H^p_{m_x}((M \otimes_z \mathcal{L}_{1y})[e-r] \otimes_y \mathcal{L}_{2x}[d]) \arrow[r, "(-1)^{re}"] & H^p_{m_x}(M[-r] \otimes_z \mathcal{L}_{1y}[e] \otimes_y \mathcal{L}_{2x}[d])
\end{tikzcd}
\tag{49}
$$

of (49) and then its right column gives the map α_2 of (47). In particular, the upper subdiagram in (47) is the same as (48) + (49), and hence commutes.

For the lower rectangle in (47), upon "canceling off" $H^p_{m_x}$ and $\otimes_y \mathcal{L}_{2x}[d]$ from each object, we reduce to showing that the outer border of the following diagram (of derived-category maps) commutes.

$$\begin{tikzcd}
(E_{\Delta_y}\mathbf{R}\varGamma'_y\mathcal{N}^\bullet)_y & (\mathbf{R}\varGamma'_y\mathcal{N}^\bullet)_y \arrow[l, "QE_{\Delta_y}\cong 1"'] \\
(H^q_y\mathbf{R}\varGamma'_y\mathcal{N}^\bullet)[-q] \arrow[u, "\text{(10) and truncation}"] & \mathbf{R}\varGamma_y\,\mathbf{R}\varGamma'_y\mathcal{N}^\bullet \arrow[l, "\text{truncation}"'] \arrow[u, "\text{canonical}"'] \\
(H^q_{m_y}\mathcal{N}^\bullet_y)[-q] \arrow[u, "(6)"'] & \mathbf{R}\varGamma_{m_y}\mathcal{N}^\bullet_y \arrow[l, "\text{truncation}"'] \arrow[u, "(6)"']
\end{tikzcd}$$

By 3.3.4, the upper rectangle commutes while the lower one commutes for functorial reasons. This completes the proof that (47), and hence (44), commutes.

The commutativity of the diagrams in (44), (43) and (42) proves that all the four subdiagrams of (41) commute. We have therefore shown that (40) commutes, thereby proving Proposition 5.2.1.

6. Closed immersions and base change

Having tackled the case of smooth maps in §5, we now look at the category of closed immersions. For a closed immersion $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathbb{F}$, the functor $f^\flat$ defined in §6.1 below provides the other concrete formula for Cousin complexes. As in §5, after relating $f^\flat$ to its punctual version, we describe a comparison isomorphism for the case of composition of two closed immersions, and give the corresponding punctual description. The proofs are straightforward in this case. In §6.2 we take up the situation of a fibered product of a smooth map and a closed immersion. The main result there is Proposition 6.2.3.

6.1. Closed Immersions. Let $h\colon (\mathcal{X}, \Delta_1) \to (\mathcal{Y}, \Delta)$ be a closed immersion in $\mathbb{F}_c$. For any complex $\mathcal{G}^\bullet$ on $\mathcal{Y}$ we define a complex $h^\flat\mathcal{G}^\bullet$ as follows

$$h^\flat\mathcal{G}^\bullet := h^{-1}\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(h_*\mathcal{O}_\mathcal{X}, \mathcal{G}^\bullet).$$

PROPOSITION 6.1.1. *Let notation and conditions be as above.*

(i) *The functor $h^\flat$ takes Cousin complexes to Cousin complexes and furthermore $h^\flat\mathrm{Coz}_\Delta(\mathcal{Y}) \subset \mathrm{Coz}_{\Delta_1}(\mathcal{X})$.*
(ii) *The functor h_* maps $\mathrm{Coz}_{\Delta_1}(\mathcal{X})$ to $\mathrm{Coz}_\Delta(\mathcal{Y})$ and furthermore there are natural isomorphisms $h^\flat h_* \xrightarrow{\sim} h^{-1}h_* = 1_\mathbf{C}$ where $\mathbf{C} = \mathrm{Coz}_{\Delta_1}(\mathcal{X})$.*

PROOF. Let $\mathcal{M}^\bullet$ be a Δ-Cousin complex on $\mathcal{Y}$. Fix $p \in \mathbb{Z}$. By definition, there is a natural decomposition $\mathcal{M}^p \cong \oplus_y i_y M_y$ where y ranges over points in $\mathcal{Y}$ such that $\Delta(y) = p$ and M_y is an $\mathcal{O}_{\mathcal{Y},y}$-module. Since $\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(h_*\mathcal{O}_\mathcal{X}, -)$ commutes with direct sums (because $h_*\mathcal{O}_\mathcal{X}$ is a coherent $\mathcal{O}_\mathcal{Y}$-module) therefore there is a canonical decomposition

$$(h^\flat\mathcal{M}^\bullet)^p \cong \oplus_y\, h^{-1}\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(h_*\mathcal{O}_\mathcal{X}, i_y M_y).$$

It suffices to consider only those $y \in \mathcal{Y}$ which have a preimage in $\mathcal{X}$. Note that for any sheaf $\mathcal{F}$ on $\mathcal{Y}$, there is a natural isomorphism

$$\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, i_y M_y) \xrightarrow{\sim} i_y\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{F}_y, M_y)$$

because, for any open neighborhood $\mathcal{V}$ of y, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{V}}(\mathcal{F}\big|_\mathcal{V}, i_y M_y\big|_\mathcal{V}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{F}_y, M_y).$$

Therefore, for $x \in \mathcal{X}$ and $y = h(x)$ there are natural isomorphisms

$$\begin{aligned} h^{-1}\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(h_*\mathcal{O}_\mathcal{X}, i_y M_y) &\xrightarrow{\sim} h^{-1}i_y\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}((h_*\mathcal{O}_\mathcal{X})_y, M_y) \\ &\xrightarrow{\sim} i_x\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, M_y). \end{aligned}$$

Thus $h^\flat\mathcal{M}^\bullet$ is a Δ_1-Cousin complex on $\mathcal{X}$ with $(h^\flat\mathcal{M}^\bullet)(x) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, M_y)$ as Δ_1, by definition, is the restriction of Δ to $\mathcal{X}$. If M_y is a zero-dimensional $\mathcal{O}_{\mathcal{Y},y}$-module then $\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, M_y)$ is a zero-dimensional $\mathcal{O}_{\mathcal{X},x}$-module. Therefore, by 2.3.5, $h^\flat\mathrm{Coz}_\Delta(\mathcal{Y}) \subset \mathrm{Coz}_{\Delta_1}(\mathcal{X})$.

With x, y as before, for any $\mathcal{O}_{\mathcal{X},x}$-module N, there is a natural isomorphism $h_*i_xN \cong i_yN$. Since h_* commutes with direct sums it follows that h_* takes Cousin complexes to Cousin complexes. Furthermore, if N is a zero-dimensional as an $\mathcal{O}_{\mathcal{X},x}$-module then it is also zero-dimensional as an $\mathcal{O}_{\mathcal{Y},y}$-module. By 2.3.5 we see that h_* maps $\mathrm{Coz}_{\Delta_1}(\mathcal{X})$ to $\mathrm{Coz}_\Delta(\mathcal{Y})$. The remaining assertions in (ii) hold more generally for arbitrary complexes on $\mathcal{X}$. □

From the proof of 6.1.1 we obtain a canonical isomorphism

$$(h^\flat \mathcal{M}^\bullet)(x) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, \mathcal{M}^\bullet(y)) \tag{50}$$

which can also be described as follows. Set $p = \Delta_1(x)$. Then (50) equals the map in degree p induced by the following map of complexes

$$(h^\flat \mathcal{M}^\bullet)_x = (\mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(h_*\mathcal{O}_\mathcal{X}, \mathcal{M}^\bullet))_y \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x}, \mathcal{M}^\bullet_y). \quad \text{(cf. 5.1.1)}$$

Let $(\mathcal{X}, \Delta_\mathcal{X}) \xrightarrow{f} (\mathcal{Y}, \Delta_\mathcal{Y}) \xrightarrow{g} (\mathcal{Z}, \Delta_\mathcal{Z})$ be closed immersions in $\mathbb{F}_\mathrm{c}$. Let $\mathcal{G}^\bullet$ be a complex on $\mathcal{Z}$. Then there is a natural isomorphism

$$f^\flat g^\flat \mathcal{G}^\bullet \longrightarrow (gf)^\flat \mathcal{G}^\bullet \tag{51}$$

defined as follows. Let $\mathcal{I}, \mathcal{J}$ denote the ideals in $\mathcal{O}_\mathcal{Z}$ that define $\mathcal{X}$ and $\mathcal{Y}$ respectively. We make the identification $g_*\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{Z}/\mathcal{J}$, $(gf)_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Z}/\mathcal{I}$, $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}/\mathcal{I}\mathcal{O}_\mathcal{Y}$. We define (51) by the following sequence of obvious natural maps

$$\begin{aligned} f^\flat g^\flat \mathcal{G}^\bullet &= f^{-1}\mathcal{H}om_\mathcal{Y}(\mathcal{O}_\mathcal{Y}/\mathcal{I}\mathcal{O}_\mathcal{Y}, g^{-1}\mathcal{H}om_\mathcal{Z}(\mathcal{O}_\mathcal{Z}/\mathcal{J}, \mathcal{G}^\bullet)) \\ &\xrightarrow{\ \sim\ } f^{-1}g^{-1}\mathcal{H}om_\mathcal{Z}(\mathcal{O}_\mathcal{Z}/\mathcal{I}, \mathcal{H}om_\mathcal{Z}(\mathcal{O}_\mathcal{Z}/\mathcal{J}, \mathcal{G}^\bullet)) \\ &\xrightarrow{\ \sim\ } f^{-1}g^{-1}\mathcal{H}om_\mathcal{Z}(\mathcal{O}_\mathcal{Z}/\mathcal{I}, \mathcal{G}^\bullet) = (gf)^\flat \mathcal{G}^\bullet. \end{aligned}$$

Let x be a point in $\mathcal{X}$. Set $y = f(x), z = g(y)$. Let A, B, C denote the local rings $\mathcal{O}_{\mathcal{Z},z}, \mathcal{O}_{\mathcal{Y},y}, \mathcal{O}_{\mathcal{X},x}$ respectively. Let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}_{\Delta_\mathcal{Z}}(\mathcal{Z})$. Set $M = \mathcal{M}^\bullet(z)$.

Proposition 6.1.2. *Under the above conditions, the following diagram commutes.*

$$\begin{tikzcd} (f^\flat g^\flat \mathcal{M}^\bullet)(x) \arrow[r, "(51)"] \arrow[d, "\text{using (50)}"'] & ((gf)^\flat \mathcal{M}^\bullet)(x) \arrow[dd, "\text{using (50)}"] \\ \mathrm{Hom}_B(C, (g^\flat \mathcal{M}^\bullet)(y)) \arrow[d, "\text{induced by (50)}"'] & \\ \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M)) \arrow[r, "(23)"] & \mathrm{Hom}_A(C, M) \end{tikzcd}$$

Proof. Set $p = \Delta_\mathcal{X}(x)$. Then the diagram in question is obtained from the degree p part of the following diagram of complexes whose vertical maps are given by the usual identification of stalks.

$$\begin{tikzcd} (f^{-1}\mathcal{H}om_\mathcal{Y}(\mathcal{O}_\mathcal{Y}/\mathcal{I}\mathcal{O}_\mathcal{Y}, g^{-1}\mathcal{H}om_\mathcal{Z}(\mathcal{O}_\mathcal{Z}/\mathcal{J}, \mathcal{M}^\bullet)))_x \arrow[r, "(51)_x"] \arrow[d, "\text{canonical}"'] & (f^{-1}g^{-1}\mathcal{H}om_\mathcal{Z}(\mathcal{O}_\mathcal{Z}/\mathcal{I}, \mathcal{M}^\bullet))_x \arrow[d, "\text{canonical}"'] \\ \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, \mathcal{M}^\bullet_z)) \arrow[r, "(23)"] & \mathrm{Hom}_A(C, \mathcal{M}^\bullet_z) \end{tikzcd}$$

Commutativity of the above diagram follows easily. □

6.2. A Fibered Product. In this subsection we are concerned with the situation of the following diagram in $\mathbb{F}_\mathrm{c}$ which is cartesian, so that $\mathcal{W} = \mathcal{X} \times_\mathcal{Y} \mathcal{Z}$,

$$\begin{tikzcd} (\mathcal{W}, \Delta_\mathcal{W}) \arrow[r, "g"] \arrow[d, "j"'] & (\mathcal{Z}, \Delta_\mathcal{Z}) \arrow[d, "i"] \\ (\mathcal{X}, \Delta_\mathcal{X}) \arrow[r, "f"] & (\mathcal{Y}, \Delta_\mathcal{Y}) \end{tikzcd} \tag{52}$$

and where f, g are smooth maps of constant relative dimension d and i, j are closed immersions. Set $E := E_{\Delta_{\mathcal{X}}}$ and $\overline{E} := E_{\Delta_{\mathcal{W}}}$, the respective Cousin functors on $\mathcal{X}, \mathcal{W}$. Let $\mathcal{L}^\bullet$ be a complex on $\mathcal{X}$. Set

$$\mathbb{E}_g(-) := \overline{E}\mathbf{R}\varGamma'_{\mathcal{W}}(g^*(-) \otimes_{\mathcal{W}} j^*\mathcal{L}^\bullet), \qquad \mathbb{E}_f(-) := E\mathbf{R}\varGamma'_{\mathcal{X}}(f^*(-) \otimes_{\mathcal{X}} \mathcal{L}^\bullet).$$

We use the following convention. Since j is a closed immersion, therefore the functor j_* is exact and hence we use j_* to denote the corresponding derived functor $\mathbf{R}j_*$ too. Also note that j_* maps $\mathcal{A}_{\mathrm{qc}}(\mathcal{W})$ to $\mathcal{A}_{\mathrm{qc}}(\mathcal{X})$.

LEMMA 6.2.1. *Let $\mathcal{J}$ be any coherent ideal in $\mathcal{O}_{\mathcal{X}}$. Set $\mathcal{I} = \mathcal{J}\mathcal{O}_{\mathcal{W}}$. Let Z be any closed subset of $\mathcal{X}$. Set $\overline{Z} = Z \cap \mathcal{W}$. Then for any $\mathcal{G}^\bullet \in \mathbf{D}^+(\mathcal{W})$ the following natural maps are isomorphisms.*

$$j_*\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{G}^\bullet \longrightarrow \mathbf{R}\varGamma_{\mathcal{J}}j_*\mathcal{G}^\bullet, \qquad j_*\mathbf{R}\varGamma_{\overline{Z}}\mathcal{G}^\bullet \longrightarrow \mathbf{R}\varGamma_Z j_*\mathcal{G}^\bullet.$$

PROOF. For the first isomorphism, we refer to [**2**, 5.2.8(d)]. Since j_* preserves flasqueness and since flasque sheaves are acyclic for the functors $\varGamma_{\overline{Z}}$ and $\varGamma_Z$, the second assertion holds because $j_*\varGamma_{\overline{Z}} = \varGamma_Z j_*$. $\square$

LEMMA 6.2.2. *For any $\mathcal{F}^\bullet \in \mathbf{D}^+(\mathcal{W})$ there is a natural isomorphism*

$$j_*\overline{E}\mathcal{F}^\bullet \cong Ej_*\mathcal{F}^\bullet.$$

PROOF. Recall that the Cousin functors $E, \overline{E}$ are given by the $E_1^{*,0}$ terms of the spectral sequences induced by the corresponding codimension filtrations on $\mathcal{X}, \mathcal{W}$. We prove the lemma by showing that either side of the required isomorphism is constructed from the same filtered complex on $\mathcal{X}$. For any subset $Z \subset \mathcal{X}$ we set $\overline{Z} = Z \cap \mathcal{W}$. Set

$$Z^p := \{x \in \mathcal{X} \mid \Delta(x) = p\} \qquad \text{and} \qquad \overline{Z}^p := Z^p \cap \mathcal{W}.$$

Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. By definition, $\overline{E}\mathcal{F}^\bullet$ is constructed using the filtration $\{\varGamma_{\overline{Z}^p}\mathcal{I}^\bullet\}_{p\in\mathbb{Z}}$ of $\mathcal{I}^\bullet$. Since j_* is exact $j_*\overline{E}\mathcal{F}^\bullet$ can be constructed using the filtration $\{j_*\varGamma_{\overline{Z}^p}\mathcal{I}^\bullet\}_{p\in\mathbb{Z}}$ of $j_*\mathcal{I}^\bullet$. On the other hand note that $j_*\mathcal{I}^\bullet$ consists of flasque sheaves (which are acyclic for the functors $\varGamma_{Z^p}$). By exactness of j_*, the natural map $j_*\mathcal{F}^\bullet \to j_*\mathcal{I}^\bullet$ is a quasi-isomorphism and hence a flasque resolution. Therefore $Ej_*\mathcal{F}^\bullet$ can be computed using the filtration $\{\varGamma_{Z^p}j_*\mathcal{I}^\bullet\}_{p\in\mathbb{Z}}$ of $j_*\mathcal{I}^\bullet$. Since for any subset $Z \subset \mathcal{X}$ we have $j_*\varGamma_{\overline{Z}} = \varGamma_Z j_*$, it follows that $\{j_*\varGamma_{\overline{Z}^p}\mathcal{I}^\bullet\}_{p\in\mathbb{Z}}$ and $\{\varGamma_{Z^p}j_*\mathcal{I}^\bullet\}_{p\in\mathbb{Z}}$ define the same filtration of $j_*\mathcal{I}^\bullet$ and hence the result follows. $\square$

For any complex $\mathcal{G}^\bullet$ on $\mathcal{Z}$ there are natural isomorphisms described below.

$$\begin{aligned} j_*(g^*\mathcal{G}^\bullet \otimes_{\mathcal{W}} j^*\mathcal{L}^\bullet) \xrightarrow{\alpha} j_*g^*\mathcal{G}^\bullet \otimes_{\mathcal{X}} j_*j^*\mathcal{L}^\bullet &\xrightarrow{\beta} f^*i_*\mathcal{G}^\bullet \otimes_{\mathcal{X}} j_*j^*\mathcal{L}^\bullet \\ &\xleftarrow{\gamma} f^*i_*\mathcal{G}^\bullet \otimes_{\mathcal{X}} \mathcal{L}^\bullet \end{aligned} \tag{53}$$

Since j is a closed immersion, j_* distributes over the tensor product and thus we obtain the isomorphism α. For β we use the base-change isomorphism $j_*g^* \cong f^*i_*$. The canonical map γ is an isomorphism because $f^*i_*\mathcal{G}^\bullet$ is supported on $\mathcal{W}$.

Let $\mathcal{F}^\bullet$ be a complex in $\mathbf{D}^+_{\mathrm{qc}}(\mathcal{W})$. Then there are natural isomorphisms

$$j_*\overline{E}\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}^\bullet \xrightarrow{\sim} Ej_*\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}^\bullet \xrightarrow{\sim} E\mathbf{R}\varGamma'_{\mathcal{X}}j_*\mathcal{F}^\bullet \tag{54}$$

where the first one is obtained using the isomorphism constructed in 6.2.2 and the second one using 6.2.1. Therefore, for any $\mathcal{G}^\bullet \in \mathbf{D}^+_{\mathrm{qc}}(\mathcal{Z})$, there are natural isomorphisms

$$\begin{aligned} j_*\mathbb{E}_g\mathcal{G}^\bullet = j_*\overline{E}\mathbf{R}\varGamma'_{\mathcal{W}}(g^*\mathcal{G}^\bullet \otimes_{\mathcal{W}} j^*\mathcal{L}^\bullet) &\xrightarrow{(54)} E\mathbf{R}\varGamma'_{\mathcal{X}} j_*(g^*\mathcal{G}^\bullet \otimes_{\mathcal{W}} j^*\mathcal{L}^\bullet) \\ &\xrightarrow{(53)} E\mathbf{R}\varGamma'_{\mathcal{X}}(f^*i_*\mathcal{G}^\bullet \otimes_{\mathcal{X}} \mathcal{L}^\bullet) = \mathbb{E}_f i_*\mathcal{G}^\bullet. \end{aligned}$$

In particular, applying j^{-1} results in an isomorphism

$$\mathbb{E}_g\mathcal{G}^\bullet \xrightarrow{\ \sim\ } j^{-1}\mathbb{E}_f i_*\mathcal{G}^\bullet. \tag{55}$$

Let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Then there is a sequence of natural maps

$$\mathbb{E}_g i^\flat\mathcal{M}^\bullet \xrightarrow{\ \sim\ } j^{-1}\mathbb{E}_f i_* i^\flat\mathcal{M}^\bullet \leftarrow j^\flat\mathbb{E}_f i_* i^\flat\mathcal{M}^\bullet \to j^\flat\mathbb{E}_f\mathcal{M}^\bullet, \tag{56}$$

where the first map is the isomorphism from (55), the second map is induced by the natural map $j^\flat \to j^{-1}$ (and is readily seen to be an isomorphism) and the third is induced by the natural map $i_*i^\flat \to 1$.

Let w be a point in $\mathcal{W}$ and let x, y, z denote the corresponding images in $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$. Set $p = \Delta_1(x), q = \Delta_2(y)$. Since the maps i, j in (52) are closed immersions, therefore $\Delta_3(z) = q$ and $\Delta_4(w) = p$. Set $p_1 := p-q+d$. By 2.6.10 the relative dimension of $\mathcal{O}_{\mathcal{X},x}/\mathcal{O}_{\mathcal{Y},y}$ and $\mathcal{O}_{\mathcal{Z},z}/\mathcal{O}_{\mathcal{W},w}$ is p_1. Set $M := \mathcal{M}^\bullet(y)$, $N := \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{Z},z}, M)$.

Proposition 6.2.3. *Let notation be as above. Assume that $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_2}(\mathcal{Y})$ and that $\mathcal{L}^\bullet = \mathcal{L}[d]$ where $\mathcal{L}$ is a quasi-coherent flat $\mathcal{O}_{\mathcal{X}}$-module and d is the relative dimension of f, g in* (52). *Then the natural maps in* (56) *are isomorphisms and the following diagram commutes.*

$$\begin{tikzcd}
(\mathbb{E}_g i^\flat\mathcal{M}^\bullet)(w) \arrow[r, "(56)"] \arrow[d, "(38)"] & (j^\flat\mathbb{E}_f\mathcal{M}^\bullet)(w) \arrow[d, "(50)"] \\
H^{p_1}_{m_w}((i^\flat\mathcal{M}^\bullet)(z) \otimes_z (j^*\mathcal{L})_w) \arrow[d, "{6.1.1,(ii)}"] & \mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{W},w}, (\mathbb{E}_f\mathcal{M}^\bullet)(x)) \arrow[d, "(38)"] \\
H^{p_1}_{m_w}(N \otimes_z (j^*\mathcal{L})_w) \arrow[r, "(24)"] & \mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{W},w}, H^{p_1}_{m_x}(M \otimes_y \mathcal{L}_x))
\end{tikzcd}$$

Proof. All the maps involved in the diagram in question are functorial in $\mathcal{M}^\bullet$ and hence using truncation arguments, we may assume without loss of generality, that $\mathcal{M}^\bullet = i_y M[-q]$. We now proceed by expanding the diagram of the Proposition horizontally according to the definition in (56). Set $j_{w^\flat}(-) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{W},w}, -)$ and set

$$\begin{aligned} \mathcal{F}_1^\bullet &= g^*i^\flat\mathcal{M}^\bullet \otimes_{\mathcal{W}} j^*\mathcal{L}[d], & \mathcal{F}_2^\bullet &= f^*i_*i^\flat\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[d], \\ F_1 &= (i^\flat\mathcal{M}^\bullet)(z) \otimes_z (j^*\mathcal{L})_w, & F_2 &= (i_*i^\flat\mathcal{M}^\bullet)(y) \otimes_y \mathcal{L}_x. \end{aligned}$$

From (53) we obtain a natural isomorphism $j_*\mathcal{F}_1^\bullet \xrightarrow{\ \sim\ } \mathcal{F}_2$. Also, as $\mathcal{O}_{\mathcal{X},x}$-modules, there is a natural isomorphism $F_1 \xrightarrow{\ \sim\ } F_2$.

The expanded version of the diagram of the Proposition occurs in (57). The downward arrows in the upper portion are obtained using (38) or (50). The horizontal maps on the top row are given by the maps in (56). The remaining maps are the obvious natural ones.

$$
\begin{tikzcd}
(\mathbb{E}_g i^\flat \mathcal{M}^\bullet)(w) \arrow[r] \arrow[dd] & (j^{-1}\mathbb{E}_f i_* i^\flat \mathcal{M}^\bullet)(w) \arrow[d, equal] & (j^\flat \mathbb{E}_f i_* i^\flat \mathcal{M}^\bullet)(w) \arrow[l] \arrow[r] \arrow[d] & (j^\flat \mathbb{E}_f \mathcal{M}^\bullet)(w) \arrow[d] \\
{} \arrow[r, phantom, "\square_1"] & (\mathbb{E}_f i_* i^\flat \mathcal{M}^\bullet)(x) \arrow[d] & j_{w\flat}((\mathbb{E}_f i_* i^\flat \mathcal{M}^\bullet)(x)) \arrow[l] \arrow[r] \arrow[d] & j_{w\flat}((\mathbb{E}_f \mathcal{M}^\bullet)(x)) \arrow[d] \\
H^{p_1}_{m_w} F_1 \arrow[r] \arrow[d] & H^{p_1}_{m_x} F_2 \arrow[d] & j_{w\flat} H^{p_1}_{m_x} F_2 \arrow[l] \arrow[r] \arrow[d] & j_{w\flat} H^{p_1}_{m_x}(M \otimes_y \mathcal{L}_x) \arrow[d, equal] \\
H^{p_1}_{m_w}(N \otimes_z (j^*\mathcal{L})_w) \arrow[r] & H^{p_1}_{m_x}(N \otimes_y \mathcal{L}_x) & j_{w\flat} H^{p_1}_{m_x}(N \otimes_y \mathcal{L}_x) \arrow[l] \arrow[r] & j_{w\flat} H^{p_1}_{m_x}(M \otimes_y \mathcal{L}_x)
\end{tikzcd}
\tag{57}
$$

$$
\begin{tikzcd}
(j_* \overline{E}\mathbf{R}\varGamma'_{\mathcal{W}} \mathcal{F}_1^\bullet)(x) \arrow[r, "(54)"] \arrow[d, "(38a)"] \arrow[dr, phantom, "\square_2"] & (E\mathbf{R}\varGamma'_{\mathcal{X}} j_* \mathcal{F}_1^\bullet)(x) \arrow[r, "(53)"] \arrow[d, "(38a)"] & (E\mathbf{R}\varGamma'_{\mathcal{X}} \mathcal{F}_2^\bullet)(x) \arrow[d, "(38a)"] \\
H^p_{m_w}(g^*(i^\flat \mathcal{M}^\bullet) \otimes_{\mathcal{W}} j^*\mathcal{L}[d])_w \arrow[r] \arrow[d, "(38b)\text{-}(38c)"] & H^p_{m_x}(j_* \mathcal{F}_1^\bullet)_x \arrow[r] \arrow[d, phantom, "\square_3"] & H^p_{m_x}(f^*(i_* i^\flat \mathcal{M}^\bullet) \otimes_{\mathcal{X}} \mathcal{L}[d])_x \arrow[d, "(38b)\text{-}(38c)"] \\
H^p_{m_w}((i^\flat \mathcal{M}^\bullet)(z)[-q] \otimes_z (j^*\mathcal{L}[d])_w) \arrow[rr] \arrow[d, "(38d)\text{-}(38e)"] & {} & H^p_{m_x}((i_* i^\flat \mathcal{M}^\bullet)(y)[-q] \otimes_y (\mathcal{L}[d])_x) \arrow[d, "(38d)\text{-}(38e)"] \\
H^{p_1}_{m_w} F_1 \arrow[r] & H^{p_1}_{m_x} F_1 \arrow[r] & H^{p_1}_{m_x} F_2
\end{tikzcd}
\tag{58}
$$

(59)
$$
\begin{tikzcd}
(j_*\overline{E}\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}_1^\bullet)(x) \arrow[r] \arrow[d] & (Ej_*\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}_1^\bullet)(x) \arrow[r] \arrow[d] & (E\mathbf{R}\varGamma'_{\mathcal{X}}j_*\mathcal{F}_1^\bullet)(x) \arrow[d] \\
(j_*H^p_{\overline{\{w\}}}\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}_1^\bullet)_x \arrow[r] & (H^p_{\overline{\{x\}}}j_*\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}_1^\bullet)_x \arrow[r] & (H^p_{\overline{\{x\}}}\mathbf{R}\varGamma'_{\mathcal{X}}j_*\mathcal{F}_1^\bullet)_x \\
(j_*H^p_{\mathcal{J}\mathcal{O}_{\mathcal{W}}}\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}_1^\bullet)_x \arrow[r] \arrow[u] \arrow[d] & (H^p_{\mathcal{J}}j_*\mathbf{R}\varGamma'_{\mathcal{W}}\mathcal{F}_1^\bullet)_x \arrow[r] \arrow[u] & (H^p_{\mathcal{J}}\mathbf{R}\varGamma'_{\mathcal{X}}j_*\mathcal{F}_1^\bullet)_x \arrow[u] \arrow[d] \\
(j_*H^p_{\mathcal{J}\mathcal{O}_{\mathcal{W}}}\mathcal{F}_1^\bullet)_x \arrow[rr] \arrow[d] & & (H^p_{\mathcal{J}}j_*\mathcal{F}_1^\bullet)_x \arrow[d] \\
H^p_{m_w}\mathcal{F}^\bullet_{1_w} \arrow[rr] & & H^p_{m_x}(j_*\mathcal{F}_1^\bullet)_x
\end{tikzcd}
$$

(60)
$$
\begin{tikzcd}
H^p_{m_w}(g^*i^\flat\mathcal{M}^\bullet\otimes_{\mathcal{W}}j^*\mathcal{L}[d])_w \arrow[r] \arrow[d, "\text{natural}"'] & H^p_{m_w}((g^*i^\flat\mathcal{M}^\bullet)_w\otimes_w(j^*\mathcal{L}[d])_w) \arrow[r] & H^p_{m_w}((i^\flat\mathcal{M}^\bullet)(z)[-q]\otimes_z(j^*\mathcal{L}[d])_w) \arrow[dd] \\
H^p_{m_x}(j_*(g^*i^\flat\mathcal{M}^\bullet\otimes_{\mathcal{W}}j^*\mathcal{L}[d]))_x \arrow[d, "\text{via }\alpha\text{ of (53)}"'] & & \\
H^p_{m_x}(j_*g^*i^\flat\mathcal{M}^\bullet\otimes_{\mathcal{X}}j_*j^*\mathcal{L}[d])_x \arrow[r] \arrow[d, "\text{via }\beta,\gamma\text{ of (53)}"'] & H^p_{m_x}((j_*g^*i^\flat\mathcal{M}^\bullet)_x\otimes_x(j_*j^*\mathcal{L}[d])_x) \arrow[r] \arrow[d, "\text{via }\beta,\gamma\text{ of (53)}"'] \arrow[dr, phantom, "\square_4"] & H^p_{m_x}((i^\flat\mathcal{M}^\bullet)(z)[-q]\otimes_z(j^*\mathcal{L}[d])_w) \arrow[d] \\
H^p_{m_x}(f^*i_*i^\flat\mathcal{M}^\bullet\otimes_{\mathcal{X}}\mathcal{L}[d])_x \arrow[r] & H^p_{m_x}((f^*i_*i^\flat\mathcal{M}^\bullet)_x\otimes_x(\mathcal{L}[d])_x) \arrow[r] & H^p_{m_x}((i_*i^\flat\mathcal{M}^\bullet)(y)[-q]\otimes_y(\mathcal{L}[d])_x)
\end{tikzcd}
$$

Commutativity of all the rectangles in (57), except for the top left one marked as $\square_1$, is straightforward to verify. We expand $\square_1$ in (58).

The unlabeled horizontal maps in (58) are the obvious natural isomorphisms. It remains to verify commutativity of $\square_2$ and $\square_3$, since commutativity of the other rectangles is evident.

In (59), we expand $\square_2$ horizontally using (54) and vertically using (12) and 3.1.9 with the notation that $\mathcal{J}$ denotes the largest coherent ideal in $\mathcal{O}_{\mathcal{X}}$ defining the closed set $\overline{\{x\}}$. Commutativity of (59) is straightforward to verify.

An expanded and *transposed* version of $\square_3$ occurs in (60). For the horizontal maps in (60) we don't use (38b)-(38c), but instead invoke 5.1.1 (localize first and then truncate). Only commutativity of $\square_4$ needs explanation, the rest being obvious.

Set $\mathcal{N}^\bullet := i^\flat\mathcal{M}^\bullet$. Then $\mathcal{N}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Z}}}(\mathcal{Z})$ and $i_*\mathcal{N}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Now commutativity of $\square_4$, where the horizontal maps are induced by truncation, follows from the commutativity of the following diagram.

$$\begin{tikzcd}
(j_*g^*\mathcal{N}^\bullet)_x \arrow[r] \arrow[d, "\text{base change}"'] & (j_*g^*\mathcal{N}^q)_x[-q] \arrow[r] \arrow[d] & \mathcal{N}^\bullet(z)[-q] \otimes_z \mathcal{O}_{\mathcal{W},w} \arrow[d] \\
(f^*i_*\mathcal{N}^\bullet)_x \arrow[r] & (f^*i_*\mathcal{N}^q)_x[-q] \arrow[r] & (i_*\mathcal{N}^\bullet)(y)[-q] \otimes_y \mathcal{O}_{\mathcal{X},x}
\end{tikzcd}$$

We have thus shown that (57) commutes. By (24), the maps in the bottom row of (57) are isomorphisms. Since the vertical maps in (57) are isomorphisms, therefore the remaining horizontal maps are also isomorphisms. This shows that (56) is an isomorphism. $\square$

7. The retract case

The main result of this section, viz., Proposition 7.3.2 is the final key ingredient, along with the main results of the preceding two sections, needed in proving the main theorem of this paper.

The title of this section refers to the situation where the identity map on an object $(\mathcal{Y}, \Delta_{\mathcal{Y}})$ in $\mathbb{F}_c$ is factored as $(\mathcal{Y}, \Delta_{\mathcal{Y}}) \xrightarrow{i} (\mathcal{X}, \Delta_{\mathcal{X}}) \xrightarrow{h} (\mathcal{Y}, \Delta_{\mathcal{Y}})$ such that i is a closed immersion and h a smooth map of constant relative dimension. In 7.3.2 we show that the Cousin functor obtained by using 1.2.2 for h, i is isomorphic to the identity functor on $\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Residues of differential forms play a role here.

After covering some homological preliminaries in §7.1 below, we prove a general result in §7.2 which provides a partial description of the coboundary map of the Cousin complex of 1.2.2(i). Proposition 7.3.2 is then an easy consequence modulo the local ingredient involving residue maps.

7.1. Homological preliminaries. Let $\mathbf{D}$ be a triangulated category with translation functor T. Consider a commutative diagram in $\mathbf{D}$ as below whose rows and columns are assumed to be triangles in $\mathbf{D}$.

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & TA_1 \\
{\scriptstyle u_1}\downarrow & & {\scriptstyle u_2}\downarrow & & {\scriptstyle u_3}\downarrow & & {\scriptstyle Tu_1}\downarrow \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & TB_1 \\
{\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & {\scriptstyle Tv_1}\downarrow \\
C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_3} & TC_1 \\
{\scriptstyle w_1}\downarrow & & {\scriptstyle w_2}\downarrow & & {\scriptstyle w_3}\downarrow & & \\
TA_1 & \xrightarrow{T\alpha_1} & TA_2 & \xrightarrow{T\alpha_2} & TA_3 & &
\end{array} \tag{61}$$

Suppose $C_1 \cong 0 \cong A_3$. Then we obtain two ways of defining a map $C_3 \to TA_1$ as follows. Consider the following diagram.

$$\begin{array}{ccc}
B_3 & \xrightarrow{\beta_3} & TB_1 \\
{\scriptstyle v_3}\downarrow & & \uparrow{\scriptstyle Tu_1} \\
C_3 & & TA_1 \\
{\scriptstyle \gamma_2}\uparrow & & \downarrow{\scriptstyle T\alpha_1} \\
C_2 & \xrightarrow{w_2} & TA_2
\end{array} \tag{62}$$

Since $C_1 \cong 0$ hence γ_2 and Tu_1 are isomorphisms. Since $A_3 \cong 0$ hence v_3 and $T\alpha_1$ are isomorphisms. Therefore, inverting these isomorphisms, we obtain one map each from the upper and lower half of (62) viz., $(Tu_1)^{-1}\beta_3 v_3^{-1}$ and $(T\alpha_1)^{-1}w_2\gamma_2^{-1}$.

We now find a condition under which these two maps coincide. Complete the map $A_1 \xrightarrow{u_2\alpha_1} B_2$ to the following triangle (uniquely determined up to isomorphism)

$$A_1 \longrightarrow B_2 \longrightarrow X \longrightarrow TA_1 \tag{63}$$

in $\mathbf{D}$ for suitable X and maps $B_2 \to X \to TA_1$ in $\mathbf{D}$. Consider the following commutative diagram whose rows are triangles in $\mathbf{D}$. (This may be imagined as comparing the triangles in the second row and second column of (61) via a triangle constructed along the diagonal.)

$$
\begin{tikzcd}
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & TB_1 \\
A_1 \arrow[u, "u_1"] \arrow[d, "\alpha_1"'] \arrow[r, "u_2\alpha_1"] & B_2 \arrow[u, equal] \arrow[d, equal] \arrow[r] & X \arrow[r] & TA_1 \arrow[u, "Tu_1"'] \arrow[d, "T\alpha_1"'] \\
A_2 \arrow[r, "u_2"] & B_2 \arrow[r, "v_2"] & C_2 \arrow[r, "w_2"] & TA_2
\end{tikzcd}
\tag{64}
$$

From the defining axioms of triangles we conclude that there exist maps (not uniquely determined)

$$X \xrightarrow{u'} B_3 \qquad \text{and} \qquad X \xrightarrow{\alpha'} C_2$$

which, when filled in (64), make it commute. (Note that u', α' are isomorphisms since the remaining vertical maps in (64) are isomorphisms.) We therefore obtain the following diagram.

$$
\begin{tikzcd}
X \arrow[r, "u'"] \arrow[d, "\alpha'"'] & B_3 \arrow[d, "v_3"] \\
C_2 \arrow[r, "\gamma_2"] & C_3
\end{tikzcd}
\tag{65}
$$

Lemma 7.1.1. *Suppose we have a commutative diagram as in* (61) *with rows and columns as triangles in* $\mathbf{D}$ *and suppose the following conditions hold.*

(i) *We have* $C_1 \cong 0 \cong A_3$ *and hence the vertical maps in* (62) *are invertible.*
(ii) *There is a choice of maps* u', α', *(satisfying their defining conditions) for which the diagram in* (65) commutes.

Then the diagram obtained by replacing the vertical maps in (62) *by their respective inverses, commutes.*

Proof. We expand (62) in the following way

$$
\begin{tikzcd}
B_3 \arrow[r, equal] \arrow[d, "v_3"'] & B_3 \arrow[r, "\beta_3"] & TB_1 \\
C_3 & X \arrow[u, "u'"] \arrow[d, "\alpha'"'] \arrow[r] & TA_1 \arrow[u, "Tu_1"'] \arrow[d, "T\alpha_1"'] \\
C_2 \arrow[u, "\gamma_2"] \arrow[r, equal] & C_2 \arrow[r, "w_2"] & TA_2
\end{tikzcd}
$$

where the two squares on the right side are obtained from (64) and hence commute by definition of u' and α', whereas the rectangle on the left side is the same as the diagram in (65). Condition (ii) therefore implies that the above diagram commutes. $\square$

Let $\mathcal{A}$ be an abelian category. Recall that to any exact sequence of complexes $0 \to L^\bullet \to M^\bullet \to N^\bullet \to 0$ in $\mathbf{C}(\mathcal{A})$ we can associate a corresponding induced triangle $L^\bullet \to M^\bullet \to N^\bullet \to L^\bullet[1]$ in $\mathbf{D}(\mathcal{A})$ (see, e.g., [**17**, Example (1.4.4)]). This association is natural, in that, any map of short exact sequences in $\mathbf{C}(\mathcal{A})$ functorially gives rise to a corresponding map of triangles. Recall further:

Lemma 7.1.2. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any triangle T in $\mathbf{D}^+(\mathcal{A})$ there exists in $\mathbf{C}^+(\mathcal{A})$, an exact sequence of complexes of injectives given by $0 \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet \longrightarrow 0$, such that the corresponding triangle $L^\bullet \longrightarrow M^\bullet \longrightarrow N^\bullet \longrightarrow L^\bullet[1]$ is isomorphic to T.*

Proof. Let $N^\bullet$, $L^\bullet$ denote complexes of injectives that are $\mathbf{D}^+(\mathcal{A})$-isomorphic to $C^\bullet$, $A^\bullet$ respectively. The natural map $C^\bullet[-1] \to A^\bullet$ induces a map $N^\bullet[-1] \to L^\bullet$ which may also be thought of as a map in $\mathbf{K}(\mathcal{A})$ and hence can be represented by a map, say α, in $\mathcal{C}(\mathcal{A})$. Let $M^\bullet$ denote the mapping cone of α. The standard exact sequence involving $L^\bullet, M^\bullet, N^\bullet$ results in a triangle, and from the defining property of triangles we see that there is a map of triangles

$$\begin{tikzcd} C^\bullet[-1] \arrow[r] \arrow[d] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[d] \\ N^\bullet[-1] \arrow[r] & L^\bullet \arrow[r] & M^\bullet \arrow[r] & N^\bullet \end{tikzcd}$$

which is in fact an isomorphism of triangles. $\square$

Let $\mathcal{A}$ be an abelian category with enough injectives. Let

$$0 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2 \longrightarrow \Gamma_3 \longrightarrow 0$$

be a sequence of additive functors from $\mathcal{A}$ to $\mathcal{A}$ which is exact on injectives of $\mathcal{A}$. Then any complex $G^\bullet$ in $\mathbf{D}^+(\mathcal{A})$ gives rise to a triangle

$$\mathbf{R}\Gamma_1 G^\bullet \xrightarrow{\text{natural}} \mathbf{R}\Gamma_2 G^\bullet \xrightarrow{\text{natural}} \mathbf{R}\Gamma_3 G^\bullet \xrightarrow{\ ?\ } \mathbf{R}\Gamma_1 G^\bullet[1] \tag{66}$$

in the following manner. Let $G^\bullet \to I^\bullet$ be an injective resolution so that we may set $\mathbf{R}\Gamma_i G^\bullet[n] = \Gamma_i I^\bullet[n]$. Then (66) is the triangle associated to the exact sequence $0 \longrightarrow \Gamma_1 I^\bullet \longrightarrow \Gamma_2 I^\bullet \longrightarrow \Gamma_3 I^\bullet \longrightarrow 0$.

Lemma 7.1.3. *With $\mathcal{A}$ and Γ_i as above, let T be a triangle in $\mathbf{D}^+(\mathcal{A})$, say $F'^\bullet \longrightarrow G^\bullet \longrightarrow F^\bullet \longrightarrow F'^\bullet[1]$, such that $\mathbf{R}\Gamma_1 F^\bullet \cong 0 \cong \mathbf{R}\Gamma_3 F'^\bullet$. Then the two rows in the following diagram (maps being the obvious natural ones) give rise to the same map from $\mathbf{R}\Gamma_3 F^\bullet$ to $\mathbf{R}\Gamma_1 F'^\bullet[1]$.*

$$\begin{tikzcd} \mathbf{R}\Gamma_3 F^\bullet \arrow[d, equal] & \mathbf{R}\Gamma_3 G^\bullet \arrow[l, "\sim"'] \arrow[r, "?", "\textit{from } (66)"'] & \mathbf{R}\Gamma_1 G^\bullet[1] & \mathbf{R}\Gamma_1 F'^\bullet[1] \arrow[l, "\sim"'] \arrow[d, equal] \\ \mathbf{R}\Gamma_3 F^\bullet & \mathbf{R}\Gamma_2 F^\bullet \arrow[l, "\sim"'] \arrow[r] & \mathbf{R}\Gamma_2 F'^\bullet[1] & \mathbf{R}\Gamma_1 F'^\bullet[1] \arrow[l, "\sim"'] \end{tikzcd}$$

Proof. By 7.1.2 we may assume, without loss of generality, that $F^\bullet, F'^\bullet, G^\bullet$ consist $\mathcal{A}$-injectives and fit into an exact sequence $0 \to F'^\bullet \to G^\bullet \to F^\bullet \to 0$ such that the corresponding induced triangle is isomorphic to T. For $j = 1, 2, 3$,

set $A_j := \Gamma_j F'^{\bullet}$, $B_j := \Gamma_j G^{\bullet}$, $C_j := \Gamma_j F^{\bullet}$. We therefore obtain the following commutative diagram in which the rows and columns are exact sequences.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{67}$$

The rows and columns give rise to triangles in $\mathbf{D}(\mathcal{A})$ and so we obtain a commutative diagram in $\mathbf{D} = \mathbf{D}(\mathcal{A})$ as in (61). The assumption $\mathbf{R}\Gamma_1 F^{\bullet} \cong 0 \cong \mathbf{R}\Gamma_3 G^{\bullet}$ implies that A_3 and C_1 are isomorphic to 0 in $\mathbf{D}$ and hence condition (i) of Lemma 7.1.1 is satisfied. In (67) the natural map $A_1 \to B_2$ is a monomorphism in $\mathbf{C}(\mathcal{A})$. Let X denote its cokernel. It follows that the natural maps $B_2 \to B_3$ and $B_2 \to C_2$ in (67) factor through the epimorphism $B_2 \to X$ and so we obtain the following induced *commutative* diagram in $\mathbf{C}(\mathcal{A})$.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & B_3 \arrow[d] \\
C_2 \arrow[r] & C_3
\end{tikzcd}$$

From the triangle associated to the exact sequence $0 \to A_1 \to B_2 \to X \to 0$ we see that X fits into a triangle as in (63). Furthermore the maps $X \to B_3$ and $X \to C_2$ of the above diagram give a choice for u', α' in (65), i.e., make (64) commute. Since for this choice of u', α', (65) commutes therefore condition (ii) of Lemma 7.1.1 is satisfied. The desired result now follows from Lemma 7.1.1. $\square$

Let $(\mathfrak{X}, \Delta)$ be a formal scheme in $\mathbb{F}_c$ and let $\{Z^m\}_{m\in\mathbb{Z}}$ be the filtration of $\mathfrak{X}$ induced by Δ. Let $\mathcal{F}$ be a flasque sheaf on $\mathfrak{X}$. Fix an integer p. Let $x_1, \dots, x_k$ be points in $Z^p \setminus Z^{p+1}$. Let $W = \overline{\{x_1, \dots, x_k\}}$ be the closure of the set $\{x_1, \dots, x_k\}$. We claim that for any i, the canonical inclusion $\varGamma_{x_i}\mathcal{F} \hookrightarrow (\varGamma_W \mathcal{F})_{x_i}$ is surjective. To that end, let U be an open neighborhood of x_i and let s be an element of $(\varGamma_W \mathcal{F})(U)$. Let Y be the closed set generated by those generic points of $\mathrm{Supp}(s)$ that do not lie in $\overline{\{x_i\}}$. Then the restriction of s to the open set $V := U \setminus Y$ lies in $(\varGamma_{\overline{\{x_i\}}}\mathcal{F})(V)$. Thus our claim follows.

Now assume that there is a point $x' \in Z^{p+1}$ such that x' is an immediate specialization of x_i for each i. Set $W' := W \cap Z^{p+1}$. Arguing as in the previous paragraph, we see that the canonical map $\varGamma_{x'}\mathcal{F} \to (\varGamma_{W'}\mathcal{F})_{x'}$ is an isomorphism.

Next note that there is a surjective map

$$\varGamma_W\mathcal{F} \twoheadrightarrow \varGamma_{W/W'}\mathcal{F} \xrightarrow[\text{[\textbf{11}, p. 227], cf. (8)}]{\sim} \bigoplus_{j=1}^{j=k} i_{x_j}(\varGamma_{x_j}\mathcal{F}).$$

Hence, by localizing at x' (resp. x_i for any i), we obtain a surjection (resp. an isomorphism)

$$(\varGamma_W\mathcal{F})_{x'} \twoheadrightarrow \oplus_j \varGamma_{x_j}\mathcal{F}, \qquad (\text{resp. } (\varGamma_W\mathcal{F})_{x_i} \xrightarrow{\sim} \varGamma_{x_i}\mathcal{F}),$$

where the inverse of the last isomorphism is the one described in the previous paragraph.

For convenience, we shall use $\varGamma_{l:m}$ to denote the functor $\varGamma_{Z^l/Z^m}$. Consider the following diagram where all the unlabeled maps are the canonical ones and a is defined below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varGamma_{x'}\mathcal{F} & \longrightarrow & (\varGamma_W\mathcal{F})_{x'} & \longrightarrow & \oplus_j\varGamma_{x_j}\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \\ 0 & \longrightarrow & (\varGamma_{p+1:p+2}\mathcal{F})_{x'} & \longrightarrow & (\varGamma_{p:p+2}\mathcal{F})_{x'} & \longrightarrow & (\varGamma_{p:p+1}\mathcal{F})_{x'} & \longrightarrow & 0 \end{array} \tag{68}$$

We define a to be the canonical inclusion induced by the decomposition in (8):

$$\bigoplus_j \varGamma_{x_j}\mathcal{F} \subseteq \bigoplus_{\{y|\Delta(y)=p,\ y\rightsquigarrow x'\}} \varGamma_y\mathcal{F} = \Big(\bigoplus_{\{y|\Delta(y)=p\}} i_y\varGamma_y\mathcal{F}\Big)_{x'} \xleftarrow[(8)]{\sim} (\varGamma_{Z^p/Z^{p+1}}\mathcal{F})_{x'}.$$

LEMMA 7.1.4. *The diagram in* (68) *commutes and its rows are exact. In particular, for any complex $\mathcal{G}^\bullet \in \mathbf{D}(\mathcal{X})$, there is a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{j=1}^{j=k} H^p_{x_j}\mathcal{G}^\bullet & \longrightarrow & H^{p+1}_{x'}\mathcal{G}^\bullet \\ \downarrow & & \downarrow \\ (H^p_{Z^p/Z^{p+1}}\mathcal{G}^\bullet)_{x'} & \longrightarrow & (H^{p+1}_{Z^{p+1}/Z^{p+2}}\mathcal{G}^\bullet)_{x'} \end{array}$$

where the downward arrows are induced from the corresponding ones in (68) *and the horizontal maps are the usual connecting homomorphisms resulting from the exact rows.*

PROOF. Using the natural maps $\varGamma_{W'} \to \varGamma_{Z^{p+1}}$, $\varGamma_W \to \varGamma_{Z^p}$, $\varGamma_{W/W'} \to \varGamma_{Z^p/Z^{p+1}}$, we expand (68) as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varGamma_{x'}\mathcal{F} & \longrightarrow & (\varGamma_W\mathcal{F})_{x'} & \longrightarrow & \oplus_j\varGamma_{x_j}\mathcal{F} & \longrightarrow & 0 \\ & & \| & & \| & & \uparrow\wr & & \\ 0 & \longrightarrow & (\varGamma_{W'}\mathcal{F})_{x'} & \longrightarrow & (\varGamma_W\mathcal{F})_{x'} & \longrightarrow & (\varGamma_{W/W'}\mathcal{F})_{x'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\varGamma_{Z^{p+1}/Z^{p+2}}\mathcal{F})_{x'} & \longrightarrow & (\varGamma_{Z^p/Z^{p+2}}\mathcal{F})_{x'} & \longrightarrow & (\varGamma_{Z^p/Z^{p+1}}\mathcal{F})_{x'} & \longrightarrow & 0 \end{array}$$

It is clear that the above diagram commutes. The middle and the bottom rows, being localization of exact sequences, are exact and hence the top row is also exact. $\square$

Looking at the image of the vertical maps in the commutative diagram of 7.1.4 we in fact see that the following diagram of induced natural maps commutes,

$$
\begin{array}{ccc}
\bigoplus_j H^p_{x_j}\mathcal{G}^\bullet & \longrightarrow & H^{p+1}_{x'}\mathcal{G}^\bullet \\
{\scriptstyle (10)}\downarrow\wr & & {\scriptstyle (10)}\downarrow\wr \\
\bigoplus_j (E_\Delta\mathcal{G}^\bullet)(x_j) & \longrightarrow & (E_\Delta\mathcal{G}^\bullet)(x')
\end{array}
\tag{69}
$$

where the bottom row is induced by the differential of $E_\Delta\mathcal{G}^\bullet$.

7.2. Coboundary for lateral specializations. Let $f\colon \mathcal{X}\to\mathcal{Y}$ be a map in $\mathbb{F}$. We say that a specialization $x \rightsquigarrow x'$ is *f-lateral* if it is an immediate specialization and its image under f is also an immediate specialization.

Let $h\colon (\mathcal{X},\Delta_{\mathcal{X}}) \to (\mathcal{Y},\Delta_{\mathcal{Y}})$ be a smooth map in $\mathbb{F}_{\mathrm{c}}$ having constant relative dimension n. Let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}(\mathcal{Y},\Delta_{\mathcal{Y}})$, and let $\mathcal{L}$ be a quasi-coherent flat $\mathcal{O}_{\mathcal{X}}$-module. Set $\mathbb{E}(\mathcal{M}^\bullet) := E_{\Delta_{\mathcal{X}}}\mathbf{R}\varGamma'_{\mathcal{X}}(h^*\mathcal{M}^\bullet\otimes\mathcal{L}[n]) \in \mathrm{Coz}(\mathcal{X},\Delta_{\mathcal{X}})$. Our aim is to describe the differential of $\mathbb{E}(\mathcal{M}^\bullet)$ when restricted to the components corresponding to a fixed h-lateral specialization.

Let $y \rightsquigarrow y'$ be an immediate specialization in $\mathcal{Y}$. Let $x' \in h^{-1}\{y'\}$ and $x_1,\dots,x_k \in h^{-1}\{y\}$ be such that x' is an immediate specialization of x_i for each i. Set

$$q = \Delta_{\mathcal{Y}}(y),\ p = \Delta_{\mathcal{X}}(x_i),\ M = \mathcal{M}^\bullet(y),\ M' = \mathcal{M}^\bullet(y').$$

Then $\Delta_{\mathcal{Y}}(y') = q+1$ and $\Delta_{\mathcal{X}}(x') = p+1$. By 2.6.10, $p_1 := p-q+n$ is the relative dimension of $\mathcal{O}_{\mathcal{X},x_i}$ over $\mathcal{O}_{\mathcal{Y},y}$ and also of $\mathcal{O}_{\mathcal{X},x'}$ over $\mathcal{O}_{\mathcal{Y},y'}$. Let ψ denote the following map of $\mathcal{O}_{\mathcal{X},x'}$-modules

$$\bigoplus_j H^{p_1}_{m_{x_j}}(M\otimes_y\mathcal{L}_{x_j}) \xrightarrow{\ \sim\ } \bigoplus_j (\mathbb{E}\mathcal{M}^\bullet)(x_j) \longrightarrow (\mathbb{E}\mathcal{M}^\bullet)(x') \xrightarrow{\ \sim\ } H^{p_1}_{m_{x'}}(M'\otimes_{y'}\mathcal{L}_{x'}),$$

where the first and the last isomorphisms are obtained using (38) and the map in the middle is induced by the differential of $\mathbb{E}\mathcal{M}^\bullet$. Let $\partial\colon M\to M'$ denote the natural map of $\mathcal{O}_{\mathcal{Y},y'}$-modules induced by the differential of $\mathcal{M}^\bullet$. Our aim is to express ψ in terms of ∂.

Set $\mathcal{O}_{x_j} := \mathcal{O}_{\mathcal{X},x_j}$ and $\mathcal{O}_{x'} := \mathcal{O}_{\mathcal{X},x'}$. Let $W = \overline{\{x_1,\dots,x_k\}}$ be the closure of the set $\{x_1,\dots,x_k\}$. Let $\mathcal{I}$ be an open coherent ideal in $\mathcal{O}_{\mathcal{X}}$ such that $\mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}) = W$. Set $I := \mathcal{I}_{x'}$. Note that $I\mathcal{O}_{\mathcal{X},x_j} = \mathcal{I}_{x_j}$ is m_{x_j}-primary for each j. Indeed, since $\mathcal{I}$ is open, we may first go modulo a defining ideal in $\mathcal{O}_{\mathcal{X}}$ so that $\mathcal{X}$ may now be assumed to be an ordinary scheme. Then the required conclusion follows easily. In particular, the canonical map $\mathbf{R}\varGamma_{m_{x_j}}(M\otimes_y\mathcal{L}_{x_j}) \to \mathbf{R}\varGamma_{I\mathcal{O}_{x_j}}(M\otimes_y\mathcal{L}_{x_j})$ is an isomorphism.

Proposition 7.2.1. *In the above situation, consider the following diagram where μ_1,μ_2,μ_3 are maps of $\mathcal{O}_{\mathcal{X},x'}$-modules defined as follows. The map μ_1 is defined on each component via the sequence of natural maps*

$$H^{p_1}_I(M\otimes_{y'}\mathcal{L}_{x'}) \to H^{p_1}_I(M\otimes_y\mathcal{L}_{x_j}) \xrightarrow{\ \sim\ } H^{p_1}_{I\mathcal{O}_{x_j}}(M\otimes_y\mathcal{L}_{x_j}) \xleftarrow{\ \sim\ } H^{p_1}_{m_{x_j}}(M\otimes_y\mathcal{L}_{x_j}),$$

μ_2 is the one induced by the canonical inclusion $\varGamma_{m_{x'}} \to \varGamma_I$ and μ_3 is $(-1)^n$ times $H^{p_1}_I(- \otimes_{y'} \mathcal{L}_{x'})$ applied to $\partial\colon M \to M'$.

$$\begin{array}{ccc}
\bigoplus_j H^{p_1}_{m_{x_j}}(M \otimes_y \mathcal{L}_{x_j}) & \xrightarrow{\ \psi\ } & H^{p_1}_{m_{x'}}(M' \otimes_{y'} \mathcal{L}_{x'}) \\
{\scriptstyle \mu_1}\big\uparrow & & \big\downarrow{\scriptstyle \mu_2} \\
H^{p_1}_I(M \otimes_{y'} \mathcal{L}_{x'}) & \xrightarrow{\ \mu_3\ } & H^{p_1}_I(M' \otimes_{y'} \mathcal{L}_{x'})
\end{array}$$

Then μ_1, μ_2 are isomorphisms and the above diagram commutes.

Proof. Using truncation arguments as in (38b) we see that ψ depends only on the modules M, M', i.e., we may assume without loss of generality that $\mathcal{M}^\bullet$ satisfies $\mathcal{M}^j = 0$ for $j \neq q, q+1$, and $\mathcal{M}^q = i_y M$, $\mathcal{M}^{q+1} = i_y M'$. Set

$$\begin{aligned}
\mathcal{F}^\bullet &:= \mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\le q}\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[n]),\\
\mathcal{F}'^\bullet &:= \mathbf{R}\varGamma'_{\mathcal{X}}(f^*\sigma_{\ge q+1}\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[n]),\\
\mathcal{G}^\bullet &:= \mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \mathcal{L}[n]).
\end{aligned}$$

Applying $\mathbf{R}\varGamma'_{\mathcal{X}}(f^*(-) \otimes_{\mathcal{X}} \mathcal{L}[n])$ to the triangle

$$\sigma_{\ge q+1}\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet \longrightarrow \sigma_{\le q}\mathcal{M}^\bullet \longrightarrow (\sigma_{\ge q+1}\mathcal{M}^\bullet)[1] \tag{70}$$

associated to the exact sequence $0 \to \sigma_{\ge q+1}\mathcal{M}^\bullet \to \mathcal{M}^\bullet \to \sigma_{\le q}\mathcal{M}^\bullet \to 0$ results in the following triangle in $\mathbf{D}(\mathcal{X})$

$$\mathcal{F}'^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{F}^\bullet \xrightarrow{\ \gamma\ } \mathcal{F}'^\bullet[1]. \tag{71}$$

The map γ is obtained from the δ-functoriality of $\mathbf{R}\varGamma'_{\mathcal{X}}(f^*(-) \otimes_{\mathcal{X}} \mathcal{L}[n])$ (§1.4, (vi)), which in turn involves the three δ-functors, $\mathbf{R}\varGamma'_{\mathcal{X}}$, f^* and $\otimes_{\mathcal{X}}\mathcal{L}[n]$. The first two of these commute with translation by construction, and the third, as seen from the convention in §1.4, (iv), also commutes with translation.

Set $F_j := M \otimes_y \mathcal{L}_{x_j}$, $F := M \otimes_{y'} \mathcal{L}_{x'}$, and $F' := M' \otimes_{y'} \mathcal{L}_{x'}$. We now expand the diagram in the assertion of the lemma as follows.

$$\begin{array}{ccccccc}
 & & \bigoplus_j (\mathbb{E}\mathcal{M}^\bullet)(x_j) & \xrightarrow{\ \mu_0\ } & (\mathbb{E}\mathcal{M}^\bullet)(x') & & \\
 & & {\scriptstyle \mu_4}\big\downarrow & & {\scriptstyle \mu_5}\big\downarrow & & \\
\bigoplus_j H^{p_1}_{m_{x_j}} F_j & \xleftarrow{\ \mu_6\ } & \bigoplus_j H^p_{x_j}\mathcal{F}^\bullet & \xrightarrow{\ \mu_7\ } & H^{p+1}_{x'}\mathcal{F}'^\bullet & \xrightarrow{\ \mu_8\ } & H^{p_1}_{m_{x'}} F' \\
{\scriptstyle \mu_1}\big\uparrow & & {\scriptstyle \mu_9}\big\uparrow & & {\scriptstyle \mu_{10}}\big\downarrow & & {\scriptstyle \mu_2}\big\downarrow \\
H^{p_1}_I F & \xleftarrow{\ \mu_{11}\ } & (H^p_W \mathcal{F}^\bullet)_{x'} & \xrightarrow{\ \mu_{12}\ } & (H^{p+1}_W \mathcal{F}'^\bullet)_{x'} & \xrightarrow{\ \mu_{13}\ } & H^{p_1}_I F'
\end{array} \tag{72}$$

The map μ_0 is the obvious one induced by the differential of $\mathbb{E}\mathcal{M}^\bullet$. We define μ_4 componentwise as the composition of the following natural maps

$$(\mathbb{E}\mathcal{M}^\bullet)(x_j) = (E_{\Delta_1}\mathcal{G}^\bullet)(x_j) \underset{(10)}{\xrightarrow{\ \sim\ }} H^p_{x_j}\mathcal{G}^\bullet \underset{(71)}{\longrightarrow} H^p_{x_j}\mathcal{F}^\bullet.$$

Note that this amounts to using (38a) and (38b) without involving the isomorphism $H^p_{x_j}\mathbf{R}\varGamma'_{\mathcal{X}}(-) \xrightarrow{\ \sim\ } H^p_{m_{x_j}}(-)_{x_j}$ (cf. (12)). We define μ_6 by first replacing $H^p_{x_j}\mathbf{R}\varGamma'_{\mathcal{X}}(-)$ by $H^p_{m_{x_j}}(-)_{x_j}$ and then following the remaining steps in (38) so that $\mu_6\mu_4$ equals (38). The maps μ_5, μ_8 are defined in an analogous fashion. We define μ_{11}, μ_{13} by following the same steps used in defining μ_6, μ_8 respectively; for example, in case of μ_{11}, we first replace $(H^p_W\mathbf{R}\varGamma'_{\mathcal{X}}(-))_{x'}$ by $H^p_I(-)_{x'}$ using the isomorphisms

$(\mathbf{R}\varGamma_W \mathbf{R}\varGamma'_{\mathfrak{X}}(-))_{x'} \cong (\mathbf{R}\varGamma_{\mathfrak{I}}(-))_{x'} \cong \mathbf{R}\varGamma_I(-)_{x'}$ and then follow (38c)-(38e). In particular, $\mu_4, \mu_5, \mu_6, \mu_8, \mu_{11}, \mu_{13}$ are all isomorphisms. For μ_9, μ_{10} we refer to the top row of (68). We define μ_7 by

$$\bigoplus_j H^p_{x_j}\mathcal{F}^\bullet \xleftarrow[(71)]{\sim} \bigoplus_j H^p_{x_j}\mathcal{G}^\bullet \xrightarrow[\text{top row}]{7.1.4} H^{p+1}_{x'}\mathcal{G}^\bullet \xleftarrow[(71)]{\sim} H^{p+1}_{x'}\mathcal{F}'^\bullet$$

and define μ_{12} to be the unique map satisfying $\mu_{13}\mu_{12}\mu_{11}^{-1} = \mu_3$.

The rectangles on the bottom left corner and the bottom right corner of (72) commute for functorial reasons. By 3.1.11, we have $\mathbf{R}\varGamma_{x'}\mathcal{F}^\bullet \cong 0$ and $\mathbf{R}\varGamma_{x_j}\mathcal{F}'^\bullet \cong 0$ and therefore by the exactness of the top row of (68) it follows that μ_9, μ_{10} are isomorphisms. In particular, μ_1, μ_2 are isomorphisms.

Since $\psi = \mu_8\mu_5\mu_0\mu_4^{-1}\mu_6^{-1}$, it follows that the commutativity statement of the lemma is the same as proving that the outer skeleton of (72) commutes and so we reduce to checking commutativity of the middle two rectangles in (72).

Vertically expanding the topmost rectangle in (72) results in the following diagram having obvious natural isomorphisms as vertical maps.

$$\begin{tikzcd}
\bigoplus_j (\mathbb{E}\mathcal{M}^\bullet)(x_j) \arrow[r, "\text{natural}"] \arrow[d, equal] & (\mathbb{E}\mathcal{M}^\bullet)(x') \arrow[d, equal] \\
\bigoplus_j (H^p_{Z^p/Z^{p+1}}\mathcal{G}^\bullet)(x_j) \arrow[d, "\wr"'] & (H^{p+1}_{Z^{p+1}/Z^{p+2}}\mathcal{G}^\bullet)(x') \arrow[d, "\wr"] \\
\bigoplus_j H^p_{x_j}\mathcal{G}^\bullet \arrow[r, "\text{top row}", "7.1.4"'] \arrow[d, "\wr"'] & H^{p+1}_{x'}\mathcal{G}^\bullet \\
\bigoplus_j H^p_{x_j}\mathcal{F}^\bullet \arrow[r, "\mu_7"] & H^{p+1}_{x'}\mathcal{F}'^\bullet \arrow[u, "\wr"']
\end{tikzcd}$$

The left and right columns in the preceding diagram give the maps μ_4 and μ_5 respectively. The commutativity of the subdiagram on the top was recorded in (69) while the bottom rectangle commutes by definition of μ_7.

So for the lemma it only remains to check that the following subrectangle of (72) commutes.

$$\begin{tikzcd}
\bigoplus_j H^p_{x_j}\mathcal{F}^\bullet \arrow[r, "\mu_7"] & H^{p+1}_{x'}\mathcal{F}'^\bullet \arrow[d, "\mu_{10}"] \\
(H^p_W\mathcal{F}^\bullet)_{x'} \arrow[u, "\mu_9"] \arrow[r, "\mu_{12}"] & (H^{p+1}_W\mathcal{F}'^\bullet)_{x'}
\end{tikzcd} \tag{73}$$

To that end we claim that μ_{12} equals $(H^p_W(-\gamma))_{x'}$ for γ as defined in (71). Assuming the claim, checking that (73) commutes amounts to checking that the following diagram "commutes" where the top row represents μ_7 and the bottom row represents the rest of (73).

$$\begin{tikzcd}
\bigoplus_j H^p_{x_j}\mathcal{F}^\bullet \arrow[d, equal] & \bigoplus_j H^p_{x_j}\mathcal{G}^\bullet \arrow[l, "\sim"'] \arrow[r, "7.1.4", "\text{top row}"'] & H^{p+1}_{x'}\mathcal{G}^\bullet & H^{p+1}_{x'}\mathcal{F}'^\bullet \arrow[l, "\sim"'] \arrow[d, equal] \\
\bigoplus_j H^p_{x_j}\mathcal{F}^\bullet & (H^p_W\mathcal{F}^\bullet)_{x'} \arrow[l, "\sim"', "\mu_9"] \arrow[r, "(H^p_W(-\gamma))_{x'}"] & (H^{p+1}_W\mathcal{F}'^\bullet)_{x'} & H^{p+1}_{x'}\mathcal{F}'^\bullet \arrow[l, "\sim"', "\mu_{10}"]
\end{tikzcd}$$

We use 7.1.3 in this situation with $\Gamma_1 = \mathit{\Gamma}_{x'}(-)$, $\Gamma_2 = (\mathit{\Gamma}_W(-))_{x'}$ and $\Gamma_3 = \oplus_j \mathit{\Gamma}_{x_j}$. Upon applying H^p to the diagram of 7.1.3 we see that the top horizontal map in 7.1.4 is (-1) times the map labeled ? in 7.1.3 (§1.4, (ix)). The remaining maps in the preceding diagram agree with those in H^p of the diagram of 7.1.3. Therefore, by 7.1.3, (73) commutes.

Verifying the above claim on μ_{12} amounts to checking that the following diagram commutes.

$$\begin{array}{ccc} (H^p_W\mathcal{F}^\bullet)_{x'} & \xrightarrow{(H^p_W(-\gamma))_{x'}} & (H^{p+1}_W\mathcal{F}'^\bullet)_{x'} \\ {\scriptstyle\mu_{11}}\downarrow & & \downarrow{\scriptstyle\mu_{13}} \\ H^{p_1}_I(M\otimes_{y'}\mathcal{L}_{x'}) & \xrightarrow{\mu_3} & H^{p_1}_I(M'\otimes_{y'}\mathcal{L}_{x'}) \end{array} \tag{74}$$

Let us expand this diagram vertically using the definition of $\mu_{11}, \mu_{13}, \mathcal{F}^\bullet$ and $\mathcal{F}'^\bullet$.

$$\begin{array}{ccc} (H^p_W\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}(f^*\sigma_{\le q}\mathcal{M}^\bullet\otimes_{\mathcal{X}}\mathcal{L}[n]))_{x'} & \xrightarrow{(H^p_W(-\gamma))_{x'}} & (H^{p+1}_W\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}(f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes_{\mathcal{X}}\mathcal{L}[n]))_{x'} \\ \downarrow & & \downarrow \\ H^p_I(f^*\sigma_{\le q}\mathcal{M}^\bullet\otimes_{\mathcal{X}}\mathcal{L}[n])_{x'} & \xrightarrow{\mu_{14}} & H^{p+1}_I(f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes_{\mathcal{X}}\mathcal{L}[n])_{x'} \\ \Vert & & \Vert \\ H^p_I(f^*i_yM[-q]\otimes_{\mathcal{X}}\mathcal{L}[n])_{x'} & \xrightarrow{\mu_{15}} & H^{p+1}_I(f^*i_{y'}M'[-q-1]\otimes_{\mathcal{X}}\mathcal{L}[n])_{x'} \\ {\scriptstyle(38c)-(38d)}\downarrow{\scriptstyle\mathrm{sign}=(-1)^{qn}} & & {\scriptstyle\mathrm{sign}=(-1)^{qn+n}}\downarrow{\scriptstyle(38c)-(38d)} \\ H^p_I((M\otimes_{y'}\mathcal{L}_{x'})[-q+n]) & \xrightarrow{\mu_{16}} & H^{p+1}_I((M'\otimes_{y'}\mathcal{L}_{x'})[-q-1+n]) \\ {\scriptstyle(38e)}\downarrow & & \downarrow{\scriptstyle(38e)} \\ H^{p_1}_I(M\otimes_{y'}\mathcal{L}_{x'}) & \xrightarrow{\mu_3} & H^{p_1}_I(M'\otimes_{y'}\mathcal{L}_{x'}) \end{array}$$

Here μ_{14} is *minus* of $H^p_{\mathfrak{p}}(-)_{x'}$ of the composite map (where $\otimes = \otimes_{\mathcal{X}}, \mathcal{L}^\bullet = \mathcal{L}[n]$)

$$f^*\sigma_{\le q}\mathcal{M}^\bullet\otimes\mathcal{L}^\bullet \xrightarrow{\Upsilon} (f^*\sigma_{\ge q+1}\mathcal{M}^\bullet)[1]\otimes\mathcal{L}^\bullet \xrightarrow{\theta} (f^*\sigma_{\ge q+1}\mathcal{M}^\bullet\otimes\mathcal{L}^\bullet)[1]$$

where Υ is induced by applying $f^*(-)\otimes\mathcal{L}^\bullet$ to the last map in (70) while θ is obtained using the convention in §1.4, (iv), and therefore, is the identity map on the graded level. It follows from the definition of γ in (71) that the topmost rectangle commutes. Next set $\mu_{15} = \mu_{14}$ so that the second rectangle from the top commutes. Keeping in mind that $\mathcal{M}^\bullet$ is a two-term complex supported in degrees $q, q+1$, one notes that on the stalks at y' the natural map

$$i_yM = \mathcal{M}^q = H^q(\sigma_{\le q}\mathcal{M}^\bullet) \xrightarrow{\text{see (70)}} H^q((\sigma_{\ge q+1}\mathcal{M}^\bullet)[1]) = \mathcal{M}^{q+1} = i_{y'}M'$$

is precisely $-\partial$ where ∂ is the canonical $\mathcal{O}_{\mathcal{Y},y'}$-linear map $M \to M'$ induced by the differential of $\mathcal{M}^\bullet$ (cf. §1.4, (ix)). Letting μ_{16} be the obvious map induced by $(-1)^n\partial$ we see that the third rectangle from the top, after going through (38c) in the vertical maps, is obtained as $H^p_{\mathfrak{p}}$ of the following commutative diagram where

$\otimes = \otimes_{y'}, L = \mathcal{L}_{x'}$.

$$\begin{tikzcd} M[-q] \otimes L[n] \arrow[r, "\partial\otimes 1"] \arrow[d, "(-1)^{qn}"] & M'[-q-1][1] \otimes L[n] \arrow[r, "\text{no}", "\text{sign}"'] & (M'[-q-1] \otimes L[n])[1] \arrow[d, "(-1)^{qn+n}"'] \\ (M \otimes L)[-q+n] \arrow[rr, "(-1)^n\partial\otimes 1"] & & (M' \otimes L)[-q+n] \end{tikzcd}$$

The rectangle involving μ_{16} and μ_3 clearly commutes and thus (74) commutes. $\square$

COROLLARY 7.2.2. *In the situation of* 7.2.1*, assume further that there exists a sequence in I of length p_1, say, $\mathbf{t} = t_1, \dots, t_{p_1}$ such that the following natural maps are isomorphisms.*

$$H^{p_1}_I(M \otimes_{y'} \mathcal{L}_{x'}) \to H^{p_1}_{\mathbf{t}\mathcal{O}_{x'}}(M \otimes_{y'} \mathcal{L}_{x'}) \qquad H^{p_1}_I(M' \otimes_{y'} \mathcal{L}_{x'}) \to H^{p_1}_{\mathbf{t}\mathcal{O}_{x'}}(M' \otimes_{y'} \mathcal{L}_{x'})$$

Then any element of $\bigoplus_j H^{p_1}_{m_{x_j}}(M \otimes_y \mathcal{L}_{x_j})$ (resp. $H^{p_1}_{m_{x'}}(M' \otimes_{y'} \mathcal{L}_{x'})$) can be written as a sum of generalized fractions of the type

$$w = \begin{bmatrix} m \otimes l \\ t_1^{r_1}, \dots, t_{p_1}^{r_{p_1}} \end{bmatrix}, \quad m \in M, l \in \mathcal{L}_{x'}, r_i > 0$$

$$(\textit{resp. } \begin{bmatrix} m' \otimes l' \\ t_1^{r'_1}, \dots, t_{p_1}^{r'_{p_1}} \end{bmatrix}, \quad m' \in M', l' \in \mathcal{L}_{x'}, r'_i > 0)$$

and ψ of 7.2.1 *sends the element represented by w to the element represented by the fraction*

$$\begin{bmatrix} (-1)^n\partial(m) \otimes l \\ t_1^{r_1}, \dots, t_{p_1}^{r_{p_1}} \end{bmatrix}.$$

PROOF. Since μ_1, μ_2 of 7.2.1 are isomorphisms, the generalized-fraction representation holds. The description of ψ in terms of such fractions follows immediately from the commutative diagram of 7.2.1. $\square$

7.3. Application to the retract case. Let $(\mathcal{Y}, \Delta_{\mathcal{Y}}) \xrightarrow{i} (\mathcal{X}, \Delta_{\mathcal{X}}) \xrightarrow{h} (\mathcal{Y}, \Delta_{\mathcal{Y}})$ be a factorization of the identity map on $\mathcal{Y}$ in $\mathbb{F}_c$, where i is a closed immersion and h a smooth map of constant relative dimension, say n. Note that i takes any immediate specialization in $\mathcal{Y}$ to an h-lateral specialization in $\mathcal{X}$. (§7.2)

LEMMA 7.3.1. *(cf.* [**27**, Thm. 2.6]*) In the above situation, let $y \rightsquigarrow y'$ be an immediate specialization in $\mathcal{Y}$ having corresponding image $x \rightsquigarrow x'$ in $\mathcal{X}$. Then there exists an $\mathcal{O}_{\mathcal{X},x'}$-sequence $\mathbf{t} = t_1, \dots, t_n$ satisfying the following properties.*

(i) *The sequence $\mathbf{t}$ maps to a regular system of parameters in $\mathcal{O}_{\mathcal{X},x'}/m_{y'}\mathcal{O}_{\mathcal{X},x'}$ and in $\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x}$. Moreover, a basis of $(\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}})_x$ and $(\widehat{\Omega}^1_{\mathcal{X}/\mathcal{Y}})_{x'}$ is given by $dt_1, \dots, dt_n$.*

(ii) *Let $\mathcal{I}$ be the largest (open) coherent $\mathcal{O}_{\mathcal{X}}$-ideal defining $\overline{\{x\}}$ and $\mathcal{K}$ the largest (open) coherent $\mathcal{O}_{\mathcal{Y}}$-ideal defining $\overline{\{y\}}$. Then $\mathcal{K}_{y'}\mathcal{O}_{\mathcal{X},x'} + \mathbf{t}\mathcal{O}_{\mathcal{X},x'} = \mathcal{I}_{x'}$.*

PROOF. The statements are local in nature. For any open subset $\mathcal{U}$ in $\mathcal{X}$, with $\mathcal{V} = i^{-1}\mathcal{U}$ we have $h(\mathcal{U}) \subset \mathcal{V}$ and h restricts to a smooth map $\mathcal{U} \to h^{-1}\mathcal{V}$. Thus we may replace $\mathcal{X}$ by $\mathcal{U} := \mathrm{Spf}(B)$ an affine open neighborhood of x' in $\mathcal{X}$, and $\mathcal{Y}$ by $\mathcal{V} = i^{-1}\mathcal{U} = \mathrm{Spf}(A)$. Then the natural induced maps $A \xrightarrow{\phi} B \xrightarrow{\pi} A$ factor the identity map on A and ϕ is a smooth map.

By replacing $\mathcal{U}, \mathcal{V}$ by smaller open subsets if necessary such that $x' \in \mathcal{U}$ and $\mathcal{V} = i^{-1}\mathcal{U}$ remain valid, we may assume, without loss of generality, that $\widehat{\Omega}^1_{\mathcal{U}/\mathcal{V}}$ is free so that the B-module $\widehat{\Omega}^1_{B/A} \xrightarrow{\sim} \Gamma(\mathcal{U}, \widehat{\Omega}^1_{\mathcal{U}/\mathcal{V}})$ (see (5) in §2.6) is free of rank n. Let $\mathcal{I}$ be the ideal in $\mathcal{O}_{\mathcal{X}}$ defining the closed immersion i. Set $J = \ker \pi$. Then $\mathcal{J}\big|_{\mathcal{U}} = J^{\sim_B}$. By 2.5.3 and 2.6.8 there are natural isomorphisms

$$J/J^2 \xrightarrow{\sim} \widehat{\Omega}^1_{B/A} \otimes_B B/J, \qquad \mathcal{J}/\mathcal{J}^2\big|_{\mathcal{U}} \xrightarrow{\sim} i^*\widehat{\Omega}^1_{\mathcal{U}/\mathcal{V}}.$$

By the Nakayama lemma it follows that $\mathcal{J}_{x'} = J\mathcal{O}_{\mathcal{X},x'}$ is generated by n elements. These elements may be assumed to be images of sections of $\mathcal{J}$ over a suitably small open affine neighborhood of x'. Again, by shrinking $\mathcal{U}, \mathcal{V}$ if necessary we may assume that J is generated over B by n elements. Let $\mathbf{t} = \{t_1, \ldots, t_n\}$ be such a generating set. The kernel $\mathcal{J}_x$ (resp. $\mathcal{J}_{x'}$) of the natural map $\mathcal{O}_{\mathcal{X},x} \to \mathcal{O}_{\mathcal{Y},y}$ (resp. $\mathcal{O}_{\mathcal{X},x'} \to \mathcal{O}_{\mathcal{Y},y'}$) induced by i is generated by the natural image of $\mathbf{t}$. Therefore, by [**19**, Thm. 8.4], the maps (completions being along the respective maximal ideals)

$$\widehat{\mathcal{O}_{\mathcal{Y},y}}[[T_1, \ldots, T_n]] \xrightarrow{\theta} \widehat{\mathcal{O}_{\mathcal{X},x}}, \qquad \widehat{\mathcal{O}_{\mathcal{Y},y'}}[[T_1, \ldots, T_n]] \xrightarrow{\theta'} \widehat{\mathcal{O}_{\mathcal{X},x'}},$$

defined by sending T_j to t_j, are surjective. Let us verify that θ, θ' are isomorphisms. Using 2.5.3 for the maps $\mathcal{O}_{\mathcal{Y},y} \to \widehat{\mathcal{O}_{\mathcal{Y},y}}[[T_1, \ldots, T_n]] \xrightarrow{\theta} \widehat{\mathcal{O}_{\mathcal{X},x}}$ we see, by comparing ranks of the free modules in 2.5.3, that for $\mathfrak{a} = \ker \theta$, we have $\mathfrak{a}/\mathfrak{a}^2 = 0$. By the Nakayama lemma $\mathfrak{a} = 0$. A similar proof works for $\ker \theta'$. Now (i) follows by considering the image of $\mathbf{t}$ in $\mathcal{O}_{\mathcal{X},x'}$.

For (ii) note that $\mathcal{J} \subset \mathcal{I}$ and moreover $\mathcal{I}/\mathcal{J}$ is mapped to $\mathcal{K}$ under the isomorphism $(\mathcal{O}_{\mathcal{X}}/\mathcal{J})\big|_{\mathcal{Z}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}$. In particular, it holds that $\mathcal{I} = \mathcal{J} + \mathcal{K}\mathcal{O}_{\mathcal{X}}$. Since $\mathcal{J}_{x'}$ is generated by $\mathbf{t}$, (ii) results. $\square$

Let $\mathcal{M}^\bullet \in \operatorname{Coz}(\mathcal{Y}, \Delta_{\mathcal{Y}})$. Set $\mathbb{E}_h(-) := E_{\Delta_{\mathcal{X}}} \mathbf{R}\varGamma'_{\mathcal{X}}(h^*(-) \otimes \omega_h[n])$. We now consider a graded isomorphism

$$i^\flat \mathbb{E}_h \mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet \tag{75}$$

which is defined pointwise, say for $y \in \mathcal{Y}$, by the following isomorphism (where $M = \mathcal{M}^\bullet(y)$, $x = i(y)$, $q = \Delta_{\mathcal{Y}}(y) = \Delta_{\mathcal{X}}(x)$, $\omega = \omega_h$)

$$\begin{aligned}(i^\flat \mathbb{E}_h \mathcal{M}^\bullet)(y) &\xrightarrow{(50)} \operatorname{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{Y},y}, (\mathbb{E}_h \mathcal{M}^\bullet)(x)) \\ &\xrightarrow{(38)} \operatorname{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{Y},y}, H^n_{m_x}(M \otimes_y \omega_x)) \xrightarrow{(-1)^{qn} \text{ times } \widehat{(25)}} M.\end{aligned}$$

For the last isomorphism we also need 2.6.9 and the isomorphisms of (19) and (20).

Proposition 7.3.2. *The graded map in* (75) *is also a map of complexes. (cf.* [**27**, Lemma 6.13]*)*

Proof. Let $y \rightsquigarrow y'$ be an immediate specialization in $\mathcal{Y}$ and let $x \rightsquigarrow x'$ denote the corresponding image in $\mathcal{X}$. Set $q = \Delta_{\mathcal{Y}}(y) = \Delta_{\mathcal{X}}(x)$, so that $q + 1 = \Delta_{\mathcal{Y}}(y') = \Delta_{\mathcal{X}}(x')$. It suffices to prove that the following diagram, whose columns give (75),

commutes. Here ψ and ∂ have the same meaning as in 7.2.1.

$$\begin{CD}
(i^\flat\mathbb{E}_h\mathcal{M}^\bullet)(y) @>>> (i^\flat\mathbb{E}_h\mathcal{M}^\bullet)(y')\\
@V{\wr}VV @V{\wr}VV\\
\mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{O}_{\mathcal{Y},y}, H^n_{m_x}(M\otimes_y\omega_x)) @. \mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x'}}(\mathcal{O}_{\mathcal{Y},y'}, H^n_{m_{x'}}(M'\otimes_{y'}\omega_{x'}))\\
@VVV @VVV\\
H^n_{m_x}(M\otimes_y\omega_x) @>{\psi}>> H^n_{m_{x'}}(M'\otimes_{y'}\omega_{x'})\\
@V{(-1)^{qn}\times\ \mathrm{res}}V{\text{of }\widehat{(25)}}V @V{(-1)^{qn+n}\times\ \mathrm{res}}V{\text{of }\widehat{(25)}}V\\
M @>{\partial}>> M'
\end{CD}$$

The commutativity of the top rectangle is straightforward to verify. We now verify commutativity of the bottom rectangle by a chase involving generalized fractions. We use 7.2.2 in this regard. Regarding the notation in 7.2.1, in this situation, we have $W=\overline{\{x\}}$, $p_1=n$ (since $\Delta_{\mathcal{X}}(x)=\Delta_{\mathcal{Y}}(y)$), $\mathcal{L}=\omega_h$ and $\mathcal{I}$ is the largest coherent ideal defining W.

Let $\mathbf{t}$ be the sequence in $\mathcal{O}_{x'}$ obtained in 7.3.1. Let us verify that $\mathbf{t}$ satisfies the hypothesis of 7.2.2. Let $\mathcal{K}$ be as in (ii) of 7.3.1. Since $\mathcal{K}_{y'}\mathcal{O}_{\mathcal{Y},y}=\mathcal{K}_y=m_y$, the latter being the maximal ideal of $\mathcal{O}_{\mathcal{Y},y}$, and since M is m_y-torsion, we see that M is $\mathcal{K}_{y'}$-torsion as an $\mathcal{O}_{\mathcal{Y},y'}$-module. With $I=\mathcal{I}_{x'}$ as in 7.2.1, by (ii) of 7.3.1 we see that the canonical map $\mathbf{R}\varGamma_I(M\otimes_{y'}\omega_{x'})\to\mathbf{R}\varGamma_{\mathbf{t}\mathcal{O}_{x'}}(M\otimes_{y'}\omega_{x'})$ is an isomorphism. Since M' is $m_{y'}$-torsion and by 7.3.1(i), $m_{x'}=m_{y'}\mathcal{O}_{x'}+\mathbf{t}\mathcal{O}_{x'}\subseteq m_{y'}\mathcal{O}_{x'}+I\subseteq m_{x'}$, the other condition also results.

By 7.3.1(i), $dt_1\wedge dt_2\wedge\cdots\wedge dt_n$ is a generator of $\omega_{x'}$ and of ω_x (cf. definition of (25)). Now chasing the image of the fraction

$$\begin{bmatrix} m\otimes dt_1\wedge dt_2\wedge\cdots\wedge dt_n\\ t_1^{r_1},\dots,t_{p_1}^{r_{p_1}}\end{bmatrix}$$

in $H^n_{m_x}(M\otimes_y\omega_x)$ we conclude by 7.2.2. □

8. The main theorem

Having established the crucial ingredients in 5.2.1, 6.1.2, 6.2.3 and 7.3.2 we are now in a position to prove the Main Theorem stated in §1.3. The crucial results are brought together in Proposition 8.3.3 below. This enables a fairly quick proof of existence of the basic pseudofunctor $(-)^\sharp$, given in §8.4.

In §8.5 we describe the behavior of $(-)^\sharp$ with respect to translations.

Let us recall some notation discussed in §1.3. Let $(\mathscr{X}, \Delta)$ be an object in $\mathbb{F}_c$. Let $\mathrm{Coz}^0_\Delta(\mathscr{X})$ be the category consisting of graded maps of the underlying graded objects of the complexes in $\mathrm{Coz}_\Delta(\mathscr{X})$. More formally, an object $\mathcal{N}^\bullet$ in $\mathrm{Coz}^0_\Delta(\mathscr{X})$ is a $\mathbb{Z}$-graded sequence of $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$-modules such that for any $p \in \mathbb{Z}$, $\mathcal{N}^p$ lies on the p-th skeleton induced by Δ. A morphism in $\mathrm{Coz}^0_\Delta(\mathscr{X})$ is simply a $\mathbb{Z}$-graded sequence of maps in $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$. There is a canonical forgetful functor $\mathrm{Coz}_\Delta(\mathscr{X}) \to \mathrm{Coz}^0_\Delta(\mathscr{X})$ that forgets the differential on a Cousin complex. For $\mathcal{N}^\bullet \in \mathrm{Coz}_\Delta(\mathscr{X})$ we shall denote its image in $\mathrm{Coz}^0_\Delta(\mathscr{X})$ also by $\mathcal{N}^\bullet$.

Let $f\colon (\mathscr{X}, \Delta_{\mathscr{X}}) \to (\mathscr{Y}, \Delta_{\mathscr{Y}})$ be a map in $\mathbb{F}_c$. For any $\mathcal{M}^\bullet \in \mathrm{Coz}^0_{\Delta_{\mathscr{Y}}}(\mathscr{Y})$, we define an object $f^\natural\mathcal{M}^\bullet \in \mathrm{Coz}^0_{\Delta_{\mathscr{X}}}(\mathscr{X})$ by (with notation explained below)

$$(f^\natural\mathcal{M}^\bullet)(x) := (\widehat{f_x})_\sharp M, \qquad x \in \mathscr{X},$$

where, with $y = f(x)$, we have $M = \mathcal{M}^\bullet(y)$, $\widehat{f_x}$ is the induced map on the completions of the stalks $\widehat{\mathcal{O}_{\mathscr{Y},y}} \to \widehat{\mathcal{O}_{\mathscr{X},x}}$ and hence is in $\mathfrak{C}$ (§4.3) and $(-)_\sharp$ is the pseudofunctor on $\mathfrak{C}$ of 4.3.1. Via the canonical forgetful functor, we shall, by abuse of notation, let $f^\natural$ take inputs from $\mathrm{Coz}_{\Delta_{\mathscr{Y}}}(\mathscr{Y})$ too. Since $(-)_\sharp$ is a pseudofunctor on $\mathfrak{C}$, we can, in an obvious manner, make $(-)^\natural$ into a pseudofunctor on $\mathbb{F}_c$.

The key step in defining $f^\sharp\mathcal{M}^\bullet$ of our Main Theorem is to specify a differential on $f^\natural\mathcal{M}^\bullet$. We do this first for the subcategory of smooth $\mathbb{F}_c$-maps and for the subcategory of closed immersions, then paste locally via factorizations, and finally use independence of the result of pasting from the choice of factorizations (Proposition 8.3.3) to globalize.

8.1. Smooth maps. For any smooth map $f\colon (\mathscr{X}, \Delta_{\mathscr{X}}) \to (\mathscr{Y}, \Delta_{\mathscr{Y}})$ in $\mathbb{F}_c$ having constant relative dimension d and for any $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathscr{Y}}}(\mathscr{Y})$, set

$$\mathbb{E}_f\mathcal{M}^\bullet := E_{\Delta_{\mathscr{X}}}\mathbf{R}\varGamma'_{\mathscr{X}}(f^*\mathcal{M}^\bullet \otimes_{\mathscr{X}} \omega_f[d]).$$

Let x be a point in $\mathscr{X}$. Set $y = f(x)$, $p = \Delta_{\mathscr{X}}(x)$, $q = \Delta_{\mathscr{Y}}(y)$. Let $\widehat{f_x}$ be the induced map on the completion of the stalks $\widehat{\mathcal{O}_{\mathscr{Y},y}} \to \widehat{\mathcal{O}_{\mathscr{X},x}}$. Let p_1 be the relative dimension of $\widehat{f_x}$, which, by 2.6.10, equals $p - q + d$. Set $M = \mathcal{M}^\bullet(y)$. Consider the following isomorphism

$$(\mathbb{E}_f\mathcal{M}^\bullet)(x) \xrightarrow{(38)} H^{p_1}_{m_x}(M \otimes_y (\omega_f)_x) \tag{76}$$
$$\xrightarrow{\text{2.6.9 and (19)}} H^{p_1}_{\widehat{m_x}}(M \otimes_{\widehat{y}} \omega_{\widehat{f_x}}) \xrightarrow{\ \theta\ } (\widehat{f_x})_\sharp M = (f^\natural\mathcal{M}^\bullet)(x)$$

where θ is $(-1)^{(p+d)q+p}$ times the isomorphism in 4.3.1, I.(i). As x ranges over $\mathscr{X}$ we therefore obtain a graded isomorphism $\mathbb{E}_f\mathcal{M}^\bullet \xrightarrow{\sim} f^\natural\mathcal{M}^\bullet$. If f does not have constant relative dimension on $\mathscr{X}$, then we restrict to connected components of $\mathscr{X}$ and then carry out the above procedure.

In particular, there is now a natural candidate for $f^\sharp\mathcal{M}^\bullet$ (with f smooth).

The isomorphism $\mathbb{E}_f \xrightarrow{\sim} f^\natural$ given by (76) behaves well with restriction to open subsets on $\mathcal{X}$. In greater detail, let $\mathcal{U}$ be an open subset of $\mathcal{X}$ and $(\mathcal{U}, \Delta) \xrightarrow{u} (\mathcal{X}, \Delta_x)$ the corresponding open immersion. Set $f_1 = fu$. Then the following diagram commutes:

$$
\begin{tikzcd}
(\mathbb{E}_f\mathcal{M}^\bullet)\big|_{\mathcal{U}} \arrow[r] \arrow[d, "\alpha"'] & (f^\natural\mathcal{M}^\bullet)\big|_{\mathcal{U}} \arrow[d, equal] \\
\mathbb{E}_{f_1}\mathcal{M}^\bullet \arrow[r] & f_1^\natural\mathcal{M}^\bullet
\end{tikzcd}
\tag{77}
$$

where α is defined through the following sequence of obvious natural maps (with $E_{\mathcal{X}} := E_{\Delta_{\mathcal{X}}}$, $E_{\mathcal{U}} := E_\Delta$)

$$
\begin{aligned}
(\mathbb{E}_f\mathcal{M}^\bullet)\big|_{\mathcal{U}} &= \big(E_{\mathcal{X}}\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \omega_f[d])\big)\big|_{\mathcal{U}} \\
&\xrightarrow{\sim} E_{\mathcal{U}}\big((\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \omega_f[d]))\big|_{\mathcal{U}}\big) \\
&\xrightarrow{\sim} E_{\mathcal{U}}\mathbf{R}\varGamma'_{\mathcal{U}}\big((f^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \omega_f[d])\big|_{\mathcal{U}}\big) \\
&\xrightarrow{\sim} E_{\mathcal{U}}\mathbf{R}\varGamma'_{\mathcal{U}}(f_1^*\mathcal{M}^\bullet \otimes_{\mathcal{U}} \omega_f\big|_{\mathcal{U}}[d]) \\
&\xrightarrow{\sim} E_{\mathcal{U}}\mathbf{R}\varGamma'_{\mathcal{U}}(f_1^*\mathcal{M}^\bullet \otimes_{\mathcal{U}} \omega_{f_1}[d]) = \mathbb{E}_{f_1}\mathcal{M}^\bullet.
\end{aligned}
$$

The commutativity of (77) is verified punctually as follows. Let x, y, p, M etc., be as before. Expanding the horizontal maps we obtain the following diagram where it suffices to verify that the rectangle on the left commutes.

$$
\begin{tikzcd}
(\mathbb{E}_f\mathcal{M}^\bullet)(x) \arrow[r, "(38a)-(38c)"] \arrow[d, "\alpha(x)"'] & H^p_{m_x}(M[-q] \otimes_y \omega_{f,x}[d]) \arrow[r, "\text{Rest of (76)}"] \arrow[d, equal] & (\widehat{f_x})_\sharp M \arrow[d, equal] \\
(\mathbb{E}_{f_1}\mathcal{M}^\bullet)(x) \arrow[r, "(38a)-(38c)"] & H^p_{m_x}(M[-q] \otimes_y \omega_{f,x}[d]) \arrow[r, "\text{Rest of (76)}"] & (\widehat{f_x})_\sharp M
\end{tikzcd}
$$

Expand the rectangle on the left, with the following notation:

$$
\mathcal{G}^\bullet := f^*\mathcal{M}^\bullet \otimes_{\mathcal{X}} \omega_f[d], \quad \mathcal{G}_1^\bullet := f_1^*\mathcal{M}^\bullet \otimes_{\mathcal{U}} \omega_{f_1}[d], \quad G^\bullet := M[-q] \otimes_y \omega_{f,x}[d].
$$

$$
\begin{tikzcd}
(E_{\mathcal{X}}\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}^\bullet)(x) \arrow[r] \arrow[d] & H^p_x\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}^\bullet \arrow[r] \arrow[d] & H^p_{m_x}\mathcal{G}^\bullet_x \arrow[r] \arrow[dd, equal] & H^p_{m_x}G^\bullet \arrow[ddd, equal] \\
(E_{\mathcal{U}}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}^\bullet)\big|_{\mathcal{U}})(x) \arrow[r] \arrow[d] & H^p_x((\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}^\bullet)\big|_{\mathcal{U}}) \arrow[d] & & \\
(E_{\mathcal{U}}\mathbf{R}\varGamma'_{\mathcal{U}}(\mathcal{G}^\bullet\big|_{\mathcal{U}}))(x) \arrow[r] \arrow[d] & H^p_x\mathbf{R}\varGamma'_{\mathcal{U}}(\mathcal{G}^\bullet\big|_{\mathcal{U}}) \arrow[r] \arrow[d] & H^p_{m_x}\mathcal{G}^\bullet_x \arrow[d] & \\
(E_{\mathcal{U}}\mathbf{R}\varGamma'_{\mathcal{U}}\mathcal{G}_1^\bullet)(x) \arrow[r] & H^p_x\mathbf{R}\varGamma'_{\mathcal{U}}\mathcal{G}_1^\bullet \arrow[r] & H^p_{m_x}\mathcal{G}^\bullet_{1\,x} \arrow[r] & H^p_{m_x}G^\bullet
\end{tikzcd}
$$

For the horizontal maps in the first two columns from the left, we refer to (12) of §3.2. Thus these two columns together correspond to (38a). The horizontal maps in the rightmost column correspond to (38b)-(38c). The vertical maps are the canonical ones. Commutativity of the above diagram is straightforward to check and thus (77) commutes.

The following is an immediate consequence of the commutativity of (77).

PROPOSITION 8.1.1. *Let* $f\colon (\mathcal{X},\Delta_x)\to(\mathcal{Y},\Delta_y)$ *be a smooth* $\mathbb{F}_c$*-map. Let* $\mathcal{U}\subset\mathcal{X}$ *be open and* $(\mathcal{U},\Delta)\xrightarrow{u}(\mathcal{X},\Delta_x)$ *the corresponding open immersion. If* $(-)^\sharp$ *is defined for smooth maps via* (76)*, then for any* $\mathcal{M}^\bullet\in\mathrm{Coz}_{\Delta_y}(\mathcal{Y})$*,* $(f^\sharp\mathcal{M}^\bullet)\big|_{\mathcal{U}}=(fu)^\sharp\mathcal{M}^\bullet$ *as complexes. In particular,* $u^\sharp$ *is the restriction of* $1_{\mathcal{X}}^\sharp$ *to* $\mathcal{U}$*.*

8.2. Closed immersions. Let $f\colon(\mathcal{X},\Delta_x)\to(\mathcal{Y},\Delta_y)$ be a closed immersion in $\mathbb{F}_c$. Then for any $\mathcal{M}^\bullet\in\mathrm{Coz}_{\Delta_y}(\mathcal{Y})$ and any $x\in\mathcal{X}$, with $y=f(x)$, $M=\mathcal{M}^\bullet(y)$, and $\widehat{f}_x$ having the obvious meaning as above, we use the following isomorphism

$$
\begin{aligned}
(78)\qquad (f^\flat\mathcal{M}^\bullet)(x) &\xrightarrow{(50)} \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y},y}}(\mathcal{O}_{\mathcal{X},x},M)\\
&\xrightarrow{(20)} \mathrm{Hom}_{\widehat{\mathcal{O}_{\mathcal{Y},y}}}(\widehat{\mathcal{O}_{\mathcal{X},x}},M)\xrightarrow{4.3.1,\ \mathrm{I.(ii)}}(\widehat{f}_x)_\natural M=(f^\natural\mathcal{M}^\bullet)(x)
\end{aligned}
$$

to get a graded isomorphism $f^\flat\mathcal{M}^\bullet\xrightarrow{\ \sim\ }f^\natural\mathcal{M}^\bullet$. In particular, there is now a natural candidate for $f^\sharp\mathcal{M}^\bullet$ for f a closed immersion.

The isomorphism $f^\flat\xrightarrow{\ \sim\ }f^\natural$ given by (78) commutes with restriction to open subsets in the following sense. Let $\mathcal{U}$ be an open subset of $\mathcal{Y}$. Set $\mathcal{V}:=f^{-1}\mathcal{U}$. Then the following diagram of obvious natural maps commutes.

$$
\begin{tikzcd}
(f^\flat\mathcal{M}^\bullet)\big|_{\mathcal{V}} \arrow[r] \arrow[d] & (f^\natural\mathcal{M}^\bullet)\big|_{\mathcal{V}} \arrow[d, equal]\\
(f\big|_{\mathcal{V}})^\flat(\mathcal{M}^\bullet\big|_{\mathcal{U}}) \arrow[r] & (f\big|_{\mathcal{V}})^\natural(\mathcal{M}^\bullet\big|_{\mathcal{U}})
\end{tikzcd}
$$

Commutativity of the above diagram is proved by showing it at the punctual level where it is straightforward to verify. In summary, if $(-)^\sharp$ is defined for closed immersions using (78), then $(f\big|_{\mathcal{V}})^\sharp(\mathcal{M}^\bullet\big|_{\mathcal{U}})=(f^\sharp\mathcal{M}^\bullet)\big|_{\mathcal{V}}$.

PROPOSITION 8.2.1. *Let* $f\colon(\mathcal{X},\Delta_x)\to(\mathcal{Y},\Delta_y)$ *be a map in* $\mathbb{F}_c$ *that is both a smooth map and a closed immersion. Then for any* $\mathcal{M}^\bullet\in\mathrm{Coz}_{\Delta_y}(\mathcal{Y})$*, the two natural candidates for a differential on* $f^\natural\mathcal{M}^\bullet$*, viz., the one induced by* (76) *and the one by* (78)*, agree. Moreover, if* $\mathcal{X}=\mathcal{Y}$ *and* f *is the identity map* $1_{\mathcal{X}}$*, then the graded isomorphism* $\delta_{\mathcal{X}}^\natural(\mathcal{M}^\bullet)\colon 1_{\mathcal{X}}^\natural\mathcal{M}^\bullet\to\mathcal{M}^\bullet$ *also induces the same differential.*

PROOF. First we show that f is an isomorphism from $\mathcal{X}$ onto a union of connected components of $\mathcal{Y}$. Let $\mathcal{V}=\mathrm{Spf}(A)$ be a connected affine open subset of $\mathcal{Y}$ for which $f^{-1}\mathcal{V}$ is non-empty. Since f is a closed immersion, $f^{-1}\mathcal{V}$ is an affine open subset, say $\mathcal{U}\cong\mathrm{Spf}(A/I)$, of $\mathcal{X}$. It suffices to show that $I=0$. Let $\mathfrak{a}$ be a defining ideal in A. By 2.4.5, the map $A\to A/I$ is flat and hence the same is true for $A/\mathfrak{a}\to A/(\mathfrak{a}+I)$. Hence $(\mathfrak{a}+I)/\mathfrak{a}$ is idempotent. Since $\mathrm{Spec}(A/\mathfrak{a})$ ($=\mathcal{V}$ as a topological space) is connected it follows that $(\mathfrak{a}+I)/\mathfrak{a}=0$ and hence $I\subset\mathfrak{a}$. As A is complete w.r.t. $\mathfrak{a}$, I is in the Jacobson radical of A. Flatness of $A\to A/I$ implies that I is idempotent and hence $I=0$ by the Nakayama lemma.

Thus we may assume without loss of generality that f is an isomorphism. We drop reference to the codimension functions for the rest of this proof. Now note that there are canonical isomorphisms

$$
f^*\mathcal{M}^\bullet\xrightarrow{\ \sim\ }\mathbb{E}_f\mathcal{M}^\bullet,\qquad f^*\mathcal{M}^\bullet\xrightarrow{\ \sim\ }f^{-1}\mathcal{M}^\bullet\xrightarrow{\ \sim\ }f^\flat\mathcal{M}^\bullet,
$$

where the first one is obtained by composing the isomorphisms

$$
f^*\mathcal{M}^\bullet\xrightarrow{\ \sim\ }Ef^*\mathcal{M}^\bullet\xrightarrow{\ \sim\ }E\mathbf{R}\varGamma'_{\mathcal{X}}f^*\mathcal{M}^\bullet\xrightarrow{\ \sim\ }E\mathbf{R}\varGamma'_{\mathcal{X}}(f^*\mathcal{M}^\bullet\otimes_{\mathcal{X}}\omega_f[d])=\mathbb{E}_f\mathcal{M}^\bullet
$$

which are based on the following:

- The complex $f^*\mathcal{M}^\bullet$ being Cousin, the isomorphism of 3.2.1(ii) applies.
- Since $f^*\mathcal{M}^\bullet$ consists of torsion modules, the canonical map is an isomorphism $\mathbf{R}\varGamma'_{\mathfrak{X}} f^*\mathcal{M}^\bullet \to f^*\mathcal{M}^\bullet$.
- In this situation $\omega_f[d] = \mathcal{O}_{\mathfrak{X}}[0]$. (Henceforth we shall identify $\otimes_{\mathfrak{X}}\mathcal{O}_{\mathfrak{X}}[0]$ with the identity functor.)

We claim that the following two graded isomorphisms are equal, thereby showing that (76) and (78) induce the same differential on $f^\natural\mathcal{M}^\bullet$:

$$(79)\qquad f^*\mathcal{M}^\bullet \xrightarrow{\ \sim\ } \mathbb{E}_f\mathcal{M}^\bullet \xrightarrow{(76)} f^\natural\mathcal{M}^\bullet, \qquad f^*\mathcal{M}^\bullet \xrightarrow{\ \sim\ } f^\flat\mathcal{M}^\bullet \xrightarrow{(78)} f^\natural\mathcal{M}^\bullet.$$

The final statement of the proposition then follows from 4.3.1, III.

For any $x \in \mathfrak{X}$, $\widehat{f}_x$ is an isomorphism, and hence to prove the above claim it suffices to show that, at the punctual level, via the canonical isomorphism

$$M \xrightarrow{\ \sim\ } M \otimes_y \mathcal{O}_{\mathfrak{X},x} \xrightarrow{\ \sim\ } \varGamma_x f^*\mathcal{M}^p = (f^*\mathcal{M}^\bullet)(x), \qquad (p = \Delta_{\mathfrak{X}}(x))$$

the two graded isomorphisms in (79) reduce to the (inverse of the) corresponding isomorphisms in 4.3.3. We now show this reduction for the first isomorphism in (79), leaving the other one to the reader.

Fix $x \in \mathfrak{X}$. Let $y = f(x), M = \mathcal{M}^\bullet(y)$. Consider the following diagram of natural isomorphisms whose top row is the first isomorphism in (79).

$$\begin{tikzcd}
 & (f^*\mathcal{M}^\bullet)(x) \arrow[r] \arrow[d] & (\mathbb{E}_f\mathcal{M}^\bullet)(x) \arrow[r, "(76)"] \arrow[d, "(38)"] & (f^\natural\mathcal{M}^\bullet)(x) \arrow[d, equal] \\
M \arrow[r] & M \otimes_y \mathcal{O}_{\mathfrak{X},x} \arrow[r] & H^0_{m_x}(M \otimes_y \mathcal{O}_{\mathfrak{X},x}) \arrow[r, "\text{rest of (76)}"] & \widehat{f}_{x\sharp}M
\end{tikzcd}$$

Note that all the signs involved in (38) and the rest of (76) vanish in this situation. The bottom row is inverse of the first isomorphism in 4.3.3 and therefore it suffices to show that the above diagram commutes. The rectangle on the right commutes trivially. We expand the one on the left as follows.

$$\begin{tikzcd}
(f^*\mathcal{M}^\bullet)(x) \arrow[d, equal] & (Ef^*\mathcal{M}^\bullet)(x) \arrow[l, "3.2.1(ii)"'] \arrow[d, "(10)"', "\alpha_1"] & (E\mathbf{R}\varGamma'_{\mathfrak{X}} f^*\mathcal{M}^\bullet)(x) \arrow[l] \arrow[d, "(10)"', "\alpha_2"] \\
(f^*\mathcal{M}^\bullet)(x) \arrow[d] & H^p_x f^*\mathcal{M}^\bullet \arrow[l, "3.2.1(i)[b]"'] \arrow[d, "\beta_1"] & H^p_x\mathbf{R}\varGamma'_{\mathfrak{X}} f^*\mathcal{M}^\bullet \arrow[l] \arrow[d, "\beta_2"] \\
M \otimes_y \mathcal{O}_{\mathfrak{X},x} & H^0_{m_x}(M \otimes_y \mathcal{O}_{\mathfrak{X},x}) \arrow[l] \arrow[r, equal] & H^0_{m_x}(M \otimes_y \mathcal{O}_{\mathfrak{X},x})
\end{tikzcd}$$

Only the maps β_i require explanation, the other nonlabeled maps being the obvious ones. Let β_2 be the unique map such that $\beta_2\alpha_2$ gives (38). (In other words, β_2 involves (38a)-(38e), except that (10), which is part of (38a), is taken care of by α_2.) We define β_1 by following the same steps as in β_2, the only difference being the absence of $\mathbf{R}\varGamma'_{\mathfrak{X}}$ (cf. (6), (12), where $\mathbf{R}\varGamma'_{\mathfrak{X}}$ may be dropped when $\mathcal{F}^\bullet$ consists of torsion modules).

The two rectangles on the right side commute for functorial reasons. For the one on the top left, see the proof of 3.2.1(iii) with $\mathcal{C}^\bullet = f^*\mathcal{M}^\bullet$. The one on the bottom

left expands to the following diagram, whose commutativity is easily verified.

$$
\begin{tikzcd}
(f^*\mathcal{M}^\bullet)(x) \arrow[r, equal] \arrow[d, equal] & H^p\varGamma_x f^*\mathcal{M}^\bullet \arrow[r] \arrow[d, equal] & H^p_x f^*\mathcal{M}^\bullet \arrow[d] \\
\varGamma_x f^*\mathcal{M}^p \arrow[r, equal] \arrow[d, equal] & H^p(\varGamma_x f^*\mathcal{M}^p[-p]) \arrow[r] & H^p_x(f^*\mathcal{M}^p[-p]) \arrow[d] \\
\varGamma_x f^*\mathcal{M}^p \arrow[rr, equal] \arrow[d] & & H^0_x f^*\mathcal{M}^p \arrow[d] \\
M \otimes_y \mathcal{O}_{\mathcal{X},x} & & H^0_{m_x}(M \otimes_y \mathcal{O}_{\mathcal{X},x}) \arrow[ll]
\end{tikzcd}
$$

$\square$

8.3. Factorizable maps. Consider a map f in $\mathbb{F}$ which admits a factorization of the type $f = f_1 f_2$ where f_2 is a closed immersion and f_1 a smooth map. The graded isomorphisms in (76) and (78), then give a candidate for $f^\sharp$. However since there is no unique choice for a factorization of f as a smooth map followed by a closed immersion we must first show that different factorizations do lead to the same definition of $f^\sharp$. We accomplish this in 8.3.3, the main result of this section.

Definition 8.3.1. Let h be a map in $\mathbb{F}_c$ and let $\mathbf{h} = h_1, \dots, h_n$ be a sequence of maps in $\mathbb{F}_c$ such that h factors as $h_n h_{n-1} \cdots h_1$ and each h_i is a smooth map or a closed immersion:

$$(\mathcal{X}, \Delta) \xrightarrow{h_1} (\mathcal{Z}_1, \Delta_1) \xrightarrow{h_2} \dots \xrightarrow{h_n} (\mathcal{Y}, \Delta_n).$$

For any $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_n}(\mathcal{Y})$ we define a Cousin complex in $\mathrm{Coz}_\Delta(\mathcal{X})$, to be denoted by $\mathbf{h}^\sharp\mathcal{M}^\bullet$ or $(h_1|\cdots|h_n)^\sharp\mathcal{M}^\bullet$, as follows. As a graded object we set $\mathbf{h}^\sharp\mathcal{M}^\bullet = h^\natural\mathcal{M}^\bullet$. We set the differential of $\mathbf{h}^\sharp\mathcal{M}^\bullet$ to be the one induced by $C^\natural_{\mathbf{h}} \colon h_1^\natural \cdots h_n^\natural \mathcal{M}^\bullet \xrightarrow{\sim} h^\natural\mathcal{M}^\bullet$ where $C^\natural_{\mathbf{h}}$ is the natural map derived from the pseudofunctoriality of $(-)^\natural$ and $h_1^\natural \cdots h_n^\natural\mathcal{M}^\bullet$ is equipped with a differential by iteratively using (76) and (78).

In the notation of 8.3.1, the underlying object of $\mathbf{h}^\sharp\mathcal{M}^\bullet$ depends only on the composite map h and not on the individual terms of the sequence $\mathbf{h}$. We want to show that the differential of $\mathbf{h}^\sharp\mathcal{M}^\bullet$ is also something that depends only on h. We first show this in four special cases. These cases are based on the results in 5.2.1, 6.1.2, 6.2.3 and 7.3.2. What follows is essentially a rewriting of these results in a way that incorporates $(-)^\natural$ via the isomorphisms in (76) and (78). It is important to get signs right, so we give elaborate proofs whenever necessary.

Case 1. *Suppose f, g are composable smooth maps in $\mathbb{F}_c$ so that the composition gf is also smooth. Then we claim that $(gf)^\sharp = (f|g)^\sharp$.* If f, g have constant relative dimension (and hence so does gf by 2.6.5), then it suffices to show that in the situation of §5.2, with $\mathcal{L}_1 = \omega_g$ and $\mathcal{L}_2 = \omega_f$, the following diagram commutes.

(80)
$$
\begin{tikzcd}
(\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) \arrow[r, "\text{(39) and 2.6.5}"] \arrow[dd, "\text{using}"', "(76)"] & (\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d, "\text{using (76)}"'] \\
 & (f^\natural\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d, "\text{using (76)}"'] \\
((gf)^\natural\mathcal{M}^\bullet)(x) & (f^\natural g^\natural\mathcal{M}^\bullet)(x) \arrow[l, "C^\natural_{f,g}"']
\end{tikzcd}
$$

Before we verify that (80) commutes we need to set up some notation. For any map $\phi\colon R \longrightarrow S$ of noetherian local rings that is formally smooth of relative dimension r and for any R-module N we set $\phi_{\ltimes} N := H^r_{m_R}(N \otimes_R \omega_\phi)$. Set $M = \mathcal{M}^\bullet(z)$. Now we expand (80) as follows with maps described below.

$$
\begin{tikzcd}
(\mathbb{E}_{gf}\mathcal{M}^\bullet)(x) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square_1"] & & (\mathbb{E}_f\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d] \\
(gf)_{x\ltimes}M \arrow[dd] & f_{x\ltimes}g_{y\ltimes}M \arrow[l] \arrow[d] \arrow[dr, phantom, "\square_3"] & f_{x\ltimes}((\mathbb{E}_g\mathcal{M}^\bullet)(y)) \arrow[l] \arrow[d] \\
\square_2 & \widehat{f}_{x\ltimes}g_{y\ltimes}M \arrow[d] & \widehat{f}_{x\ltimes}((\mathbb{E}_g\mathcal{M}^\bullet)(y)) \arrow[l] \arrow[dd, "\gamma_1"] \\
\widehat{(gf)}_{x\ltimes}M \arrow[dd, "\alpha"] & \widehat{f}_{x\ltimes}\widehat{g}_{y\ltimes}M \arrow[l, "\beta"'] \arrow[d, "\gamma_2"'] \arrow[r, phantom, "\square_4"] & {} \\
\square_5 & \widehat{f}_{x\sharp}\widehat{g}_{y\ltimes}M \arrow[d, "\delta_2"'] & \widehat{f}_{x\sharp}((\mathbb{E}_g\mathcal{M}^\bullet)(y)) \arrow[l] \arrow[d, equal] \\
\widehat{(gf)}_{x\sharp}M \arrow[d, equal] \arrow[dr, phantom, "\square_7"] & \widehat{f}_{x\sharp}\widehat{g}_{y\sharp}M \arrow[l] \arrow[d] \arrow[r, phantom, "\square_6"] & (f^\natural\mathbb{E}_g\mathcal{M}^\bullet)(x) \arrow[d, "\delta_1"] \\
((gf)^\natural\mathcal{M}^\bullet)(x) & (f^\natural g^\natural\mathcal{M}^\bullet)(x) \arrow[l] \arrow[r, equal] & (f^\natural g^\natural\mathcal{M}^\bullet)(x)
\end{tikzcd}
$$

The rectangle $\square_1$ is the same as the diagram in 5.2.1 via the isomorphism in 2.6.5. Consequently $\square_1$ commutes. The vertical maps in $\square_2, \square_3$ are the ones induced by going to completion (see (19)) while the remaining horizontal maps are the obvious induced ones. Consequently $\square_2, \square_3$ commute. In $\square_4$, γ_1, γ_2 are induced by the isomorphism $\widehat{f}_{x\ltimes} \xrightarrow{\sim} \widehat{f}_{x\sharp}$ defined exactly as in the map θ in (76), (i.e., $\gamma_i = (-1)^{(p+d)q+p}$ times the map induced by 4.3.1, I.(i)). The horizontal maps being the obvious ones, $\square_4$ commutes. In $\square_6$, δ_1 is the natural map induced by the isomorphism $\mathbb{E}_g\mathcal{M}^\bullet \xrightarrow{\sim} g^\natural\mathcal{M}^\bullet$ of (76). The remaining maps in $\square_6$ spell out the definition in (76), in particular, δ_2 is $(-1)^{(q+e)r+q}$ times the isomorphism given by 4.3.1, I.(i). Therefore $\square_6$ commutes. The vertical maps in $\square_7$ are equalities while the horizontal maps are obtained from the comparison isomorphism $C_\sharp^{f,g}$ and so $\square_7$ commutes. Finally in $\square_5$, α is another instance of the map θ in (76) so that α is $(-1)^{(p+d+e)r+p}$ times the isomorphism of 4.3.1, I.(i). Note that β, which is obtained from (22), is $(-1)^{(q-r)(p+d-q)}$ times the the bottom row of 4.3.1, II.(i), (because $q-r$ is the transcendence degree of the induced map of residue fields $k(z) \to k(y)$ and $p+d-q$ is the relative dimension of $\mathcal{O}_{\mathcal{X},x}$ over $\mathcal{O}_{\mathcal{Y},y}$). From the commutative diagram in 4.3.1, II.(i) and the following calculation

$$(-1)^{(p+d+e)r+p}(-1)^{(q-r)(p+d-q)} = (-1)^{(p+d)q+p}(-1)^{(q+e)r+q},$$

we conclude that $\square_5$ commutes. This proves that (80) commutes.

By first restricting to connected components, if necessary, the the condition on constant relative dimension for f, g is relaxed, thus proving Case 1 in general. As

a corollary we now have a Coz-valued pseudofunctor $(-)^\sharp$ over the subcategory of smooth maps in $\mathbb{F}_c$.

Case 2. *Suppose f, g are closed immersions in $\mathbb{F}_c$ such that the composition gf exists and hence is also a closed immersion. Then $(gf)^\sharp = (f|g)^\sharp$.* For this it suffices to show that in the situation of 6.1.2 the following diagram commutes.

$$\begin{tikzcd}
(f^\flat g^\flat \mathcal{M}^\bullet)(x) \arrow[r] \arrow[d] & ((gf)^\flat \mathcal{M}^\bullet)(x) \arrow[d] \\
(f^\natural g^\natural \mathcal{M}^\bullet)(x) \arrow[r, "C^\natural_{f,g}"] & ((gf)^\natural \mathcal{M}^\bullet)(x)
\end{tikzcd}$$

The commutativity of this diagram follows from 6.1.2 and the commutative diagram in $(\widehat{23})$. As a corollary we now have $(-)^\sharp$ as a pseudofunctor over the subcategory of closed immersions in $\mathbb{F}_c$.

Case 3. *For the diagram in* (52), *we claim that $(j|f)^\sharp = (g|i)^\sharp$.* It suffices to show that in the situation of 6.2.3, with $\mathcal{L} = \omega_f$, the following diagram commutes.

$$\begin{tikzcd}
(\mathbb{E}_g i^\flat \mathcal{M}^\bullet)(w) \arrow[r] \arrow[d] & (j^\flat \mathbb{E}_f \mathcal{M}^\bullet)(w) \arrow[d] \\
(g^\natural i^\flat \mathcal{M}^\bullet)(w) \arrow[d] & (j^\natural \mathbb{E}_f \mathcal{M}^\bullet)(w) \arrow[d] \\
(g^\natural i^\natural \mathcal{M}^\bullet)(w) \arrow[r] & (j^\natural f^\natural \mathcal{M}^\bullet)(w)
\end{tikzcd} \tag{81}$$

To prove this we expand (81) as in (82) with the following notation. For formally smooth maps of noetherian local rings we use the notation of $(-)_\ltimes$ as in Case 1. For any surjective map $\psi\colon R \longrightarrow S$ of noetherian local rings and for any R-module N we set $\psi_\rtimes N := \mathrm{Hom}_R(S, N)$. In what follows, those maps of (82) which are not specifically described are assumed to have obvious meanings.

Modulo the relation $\epsilon\epsilon^{-1} = 1$, the diagram in $\square_1$ is the same as that in 6.2.3 and hence $\square_1$ commutes. The maps in $\square_2$ are obtained from completion and the isomorphism of (24), so $\square_2$ commutes. The maps $\mu_1, \mu_2, \mu_3, \mu_4$ are induced by the isomorphisms $\widehat{g_w}_\ltimes \xrightarrow{\sim} \widehat{g_w}_\sharp$ and $\widehat{f_x}_\ltimes \xrightarrow{\sim} \widehat{f_x}_\sharp$ which, in turn, are defined exactly as θ is defined in (76) (and hence equal $(-1)^{(p+d)q+p}$ times the corresponding map induced by 4.3.1, I.(i)). We obtain ν from the pseudofunctoriality of $\sharp$. From the commutative diagram in 4.3.1, II.(iii), it follows that $\square_3$ commutes. In $\square_4$, κ_1 is induced by an application of (78) while κ_2 in $\square_5$ is induced from (76). The remaining maps in $\square_4, \square_5$ spell out the definition of the maps κ_i and hence $\square_4, \square_5$ commute. The remaining rectangles in (82) commute due to functorial reasons. Thus (82), and hence (81), commutes.

Case 4. *Suppose $\mathscr{Y} \xrightarrow{i} \mathscr{X} \xrightarrow{h} \mathscr{Y}$ is a factorization of the identity map on $\mathscr{Y}$ with i a closed immersion and h a smooth map in $\mathbb{F}$. Then $(i|h)^\sharp = (1_\mathscr{Y})^\sharp$.* For this it suffices to show that in the situation of 7.3.2, for any $y \in \mathscr{Y}$, the diagram

$$\begin{tikzcd}
(i^\flat \mathbb{E}_h \mathcal{M}^\bullet)(y) \arrow[r, "(75)"] \arrow[d, "(76)=(78)"'] & \mathcal{M}^\bullet(y) \arrow[d, "\delta^\natural_y"] \\
(i^\natural h^\natural \mathcal{M}^\bullet)(y) \arrow[r, "C^\natural_{i,h}"] & ((1_\mathscr{Y})^\natural \mathcal{M}^\bullet)(y)
\end{tikzcd} \tag{83}$$

commutes, which can be done along the same lines as the earlier cases.

$$
\begin{tikzcd}
(\mathbb{E}_g i^\flat \mathcal{M}^\bullet)(w) \arrow[rrrrr] \arrow[d] \arrow[drrrrr, phantom, "\square_1"] & & & & & (j^\flat \mathbb{E}_f \mathcal{M}^\bullet)(w) \arrow[d] \\
g_{w\ltimes}((i^\flat \mathcal{M}^\bullet)(z)) \arrow[r] \arrow[d] & g_{w\ltimes} i_{z\rtimes} M \arrow[r] \arrow[d] & j_{w\rtimes} f_{x\ltimes} M \arrow[r, "\epsilon"] \arrow[d, phantom, "\square_2"] & j_{w\rtimes} \widehat{f}_{x\ltimes} M \arrow[r, "\epsilon^{-1}"] \arrow[d] & j_{w\rtimes} f_{x\ltimes} M \arrow[r] \arrow[d] & j_{w\rtimes}((\mathbb{E}_f \mathcal{M}^\bullet)(x)) \arrow[d] \\
\widehat{g_{w\ltimes}}((i^\flat \mathcal{M}^\bullet)(z)) \arrow[r] \arrow[d, "\mu_1"] & \widehat{g_{w\ltimes}} i_{z\rtimes} M \arrow[r] \arrow[d, "\mu_2"] & \widehat{g_{w\ltimes}} \widehat{i_{z\rtimes}} M \arrow[r] \arrow[d, "\mu_3"] & \widehat{j_{w\rtimes}} \widehat{f}_{x\ltimes} M \arrow[r] \arrow[d] & \widehat{j_{w\rtimes}} f_{x\ltimes} M \arrow[r] \arrow[d] & \widehat{j_{w\rtimes}}((\mathbb{E}_f \mathcal{M}^\bullet)(x)) \arrow[d] \\
\widehat{g_{w\sharp}}((i^\flat \mathcal{M}^\bullet)(z)) \arrow[r] \arrow[d, equal] & \widehat{g_{w\sharp}} i_{z\rtimes} M \arrow[r] & \widehat{g_{w\sharp}} \widehat{i_{z\rtimes}} M \arrow[d] \arrow[r, phantom, "\square_3"] & \widehat{j_{w\sharp}} \widehat{f}_{x\ltimes} M \arrow[r] \arrow[d, "\mu_4"] & \widehat{j_{w\sharp}} f_{x\ltimes} M \arrow[r] & \widehat{j_{w\sharp}}((\mathbb{E}_f \mathcal{M}^\bullet)(x)) \arrow[d, equal] \\
(g^\natural i^\flat \mathcal{M}^\bullet)(w) \arrow[d, "\kappa_1"] \arrow[rr, phantom, "\square_4"] & & \widehat{g_{w\sharp}} \widehat{i_{z\sharp}} M \arrow[r, "\nu"] \arrow[d, equal] & \widehat{j_{w\sharp}} \widehat{f}_{x\sharp} M \arrow[d, equal] \arrow[rr, phantom, "\square_5"] & & (j^\natural \mathbb{E}_f \mathcal{M}^\bullet)(w) \arrow[d, "\kappa_2"] \\
(g^\natural i^\natural \mathcal{M}^\bullet)(w) \arrow[rr, equal] & & (g^\natural i^\natural \mathcal{M}^\bullet)(w) \arrow[r] & (j^\natural f^\natural \mathcal{M}^\bullet)(w) \arrow[rr, equal] & & (j^\natural f^\natural \mathcal{M}^\bullet)(w)
\end{tikzcd}
\tag{82}
$$

Before we put together the four cases above, we will need some basic results involving the definition in 8.3.1. For the rest of this subsection, we do not write the Cousin complexes that occur as inputs of the functors we work with. We also drop reference to the codimension functions. We call a sequence $\mathbf{f} = f_1, \ldots, f_n$ of maps in $\mathbb{F}$ *composable* if, for each i, the target of f_i equals the source of f_{i+1}. For any i, $\mathbf{f}_{\leq i}$ denotes the sequence $f_1, \ldots, f_i$ and $\mathbf{f}_{>i}$ denotes the sequence $f_{i+1}, \ldots, f_n$.

LEMMA 8.3.2. *Let $\mathbf{f} = f_1, \ldots, f_n$ be a composable sequence in $\mathbb{F}$, such that each f_i is a smooth map or a closed immersion.*

(i) *For any i, the canonical graded isomorphism*

$$(\mathbf{f}_{\leq i})^\sharp(\mathbf{f}_{>i})^\sharp = (f_i \cdots f_1)^\natural(f_n \cdots f_{i+1})^\natural \xrightarrow{\sim} (f_n \cdots f_1)^\natural = \mathbf{f}^\sharp$$

is a map of complexes.

(ii) *Suppose that for some i there exist maps g_i, g_{i+1} in $\mathbb{F}$ such that*
  (a) *$g_{i+1}g_i = f_{i+1}f_i$ and g_i, g_{i+1} are smooth maps or closed immersions;*
  (b) *$(g_i|g_{i+1})^\sharp = (f_i|f_{i+1})^\sharp$.*

If $\mathbf{g}$ denotes the (composable) sequence obtained by replacing f_i, f_{i+1} in $\mathbf{f}$ by g_i, g_{i+1} respectively, then $\mathbf{f}^\sharp = \mathbf{g}^\sharp$.

PROOF. The following diagram, whose maps are the canonical graded isomorphisms, commutes by pseudofunctoriality. Since the vertical maps define the differentials on $\mathbf{f}_{\leq i}, \mathbf{f}_{>i}$, (i) follows.

$$\begin{tikzcd}
(f_i \cdots f_1)^\natural(f_n \cdots f_{i+1})^\natural \arrow[r] \arrow[d] & (f_n \cdots f_1)^\natural \arrow[d] \\
f_1^\natural \cdots f_i^\natural f_{i+1}^\natural \cdots f_n^\natural \arrow[r, equal] & f_1^\natural \cdots f_n^\natural
\end{tikzcd}$$

For (ii), consider the following diagram, whose maps are the canonical isomorphisms induced by $(-)^\natural$.

$$\begin{tikzcd}
(f_{i+1}f_i)^\natural \arrow[r, equal] \arrow[d] & (g_{i+1}g_i)^\natural \arrow[d] \\
f_i^\natural f_{i+1}^\natural \arrow[r] & g_i^\natural g_{i+1}^\natural
\end{tikzcd}$$

By assumption, if the objects in the bottom row are equipped with a differential by using (76) and (78), then the bottom row is a map of complexes. Therefore, the analogous statement holds for the following diagram.

$$\begin{tikzcd}
(f_n \cdots f_1)^\natural \arrow[r, equal] \arrow[d] & (f_n \cdots f_1)^\natural \arrow[d] \\
f_1^\natural \cdots f_i^\natural f_{i+1}^\natural \cdots f_n^\natural \arrow[r] & f_1^\natural \cdots f_{i-1}^\natural g_i^\natural g_{i+1}^\natural f_{i+2}^\natural \cdots f_n^\natural
\end{tikzcd}$$

Thus (ii) follows. □

PROPOSITION 8.3.3. *Let $\mathcal{X} \xrightarrow{i_1} \mathcal{Z}_1 \xrightarrow{h_1} \mathcal{Y}$ and $\mathcal{X} \xrightarrow{i_2} \mathcal{Z}_2 \xrightarrow{h_2} \mathcal{Y}$ be maps in $\mathbb{F}$ such that h_j are smooth maps, i_j closed immersions and $h_1i_1 = h_2i_2$. Assume further that h_1i_1 is separated. Then $(i_1|h_1)^\sharp = (i_2|h_2)^\sharp$.*

PROOF. Consider the following diagram with obvious choices for the maps and notation.

$$
\begin{array}{ccccccc}
\mathcal{X} & \xrightarrow{d} & \mathcal{X}\times_{\mathcal{Y}}\mathcal{X} & \xrightarrow{i_1''} & \mathcal{Z}_1\times_{\mathcal{Y}}\mathcal{X} & \xrightarrow{h_1''} & \mathcal{X} \\
 & & \big\downarrow{\scriptstyle i_2''} & & \big\downarrow{\scriptstyle i_2'} & & \big\downarrow{\scriptstyle i_2} \\
 & & \mathcal{X}\times_{\mathcal{Y}}\mathcal{Z}_2 & \xrightarrow{i_1'} & \mathcal{Z}_1\times_{\mathcal{Y}}\mathcal{Z}_2 & \xrightarrow{h_1'} & \mathcal{Z}_2 \\
 & & \big\downarrow{\scriptstyle h_2''} & & \big\downarrow{\scriptstyle h_2'} & & \big\downarrow{\scriptstyle h_2} \\
 & & \mathcal{X} & \xrightarrow{i_1} & \mathcal{Z}_1 & \xrightarrow{h_1} & \mathcal{Y}
\end{array}
\tag{84}
$$

Separatedness of h_1i_1 and $h_2''i_2''$ implies that d is a closed immersion and by Case 2 and Case 4 above we conclude that $(d\,|i_2''|h_2'')^\sharp = 1^\sharp_{\mathcal{X}} = (d\,|i_1''|h_1'')^\sharp$. Therefore we obtain the following isomorphisms of complexes

$$
\begin{aligned}
1^\sharp_{\mathcal{X}}(i_1|h_1)^\sharp = (d\,|i_2''|h_2'')^\sharp(i_1|h_1)^\sharp &\xrightarrow{\alpha} (d\,|i_2''|h_2''|i_1|h_1)^\sharp \\
&\xrightarrow{\beta} (d\,|i_1''|h_1''|i_2|h_2)^\sharp \\
&\xrightarrow{\gamma} (d\,|i_1''|h_1'')^\sharp(i_2|h_2)^\sharp = 1^\sharp_{\mathcal{X}}(i_2|h_2)^\sharp
\end{aligned}
$$

where α, γ are obtained using 8.3.2, (i), while β is defined as follows. We begin from the southwest corner of (84). By Case 3 above applied to the bottom left rectangle in (84) and by 8.3.2, (ii), we may replace the subsequence h_2'', i_1 by i_1', h_2'. Proceeding in this manner for the remaining rectangles in (84) (using Cases 1, 2, 3 above and 8.3.2, (ii)), we reach the northeast corner of (84) to complete the definition of β.

Since α, β, γ are all equalities and $1^\sharp_{\mathcal{X}}$ is isomorphic to the identity functor on $\mathrm{Coz}(\mathcal{X})$ we see that $(i_1|h_1)^\sharp = (i_2|h_2)^\sharp$. □

8.4. Constructing $(-)^\sharp$ for a general map. In order to extend the preceding results for factorizable maps as in 8.3.3 to arbitrary maps in $\mathbb{F}$ we first need a localization result. For the following Lemma, we drop reference to the codimension functions and the Cousin complexes that occur as inputs to the functors involved.

LEMMA 8.4.1. *Let $\mathcal{U} \xrightarrow{i} \mathcal{Z} \xrightarrow{h} \mathcal{Y}$ be maps in $\mathbb{F}$ where i is a closed immersion and h is a smooth map. Let $\mathcal{U}'$ be an open subset of $\mathcal{U}$ so that there exists some open subset $\mathcal{Z}'$ of $\mathcal{Z}$ for which $i^{-1}\mathcal{Z}' = \mathcal{U}'$. Let $\mathcal{U}' \xrightarrow{i'} \mathcal{Z}' \xrightarrow{h'} \mathcal{Y}$ denote the induced maps. Then $(i|h)^\sharp\big|_{\mathcal{U}'} = (i'|h')^\sharp$.*

PROOF. This follows immediately from the fact that over smooth maps and closed immersions, $(-)^\sharp$ behaves well with respect to restriction to open subsets (cf. 8.1.1). □

We are now in a position to define $(-)^\sharp$ of the Main Theorem over the whole of $\mathbb{F}_c$. Let $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ be a map in $\mathbb{F}_c$. Let $\mathcal{B} = \{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ be the collection of all open subsets of $\mathcal{X}$ such that for any λ, the induced map $f_\lambda\colon \mathcal{U}_\lambda \to \mathcal{Y}$ admits a factorization $\mathcal{U}_\lambda \xrightarrow{i_\lambda} \mathcal{Z}_\lambda \xrightarrow{h_\lambda} \mathcal{Y}$ where i_λ is a closed immersion and h_λ a separated smooth map. By 2.4.4, $\mathcal{B}$ forms a basis for open sets in $\mathcal{X}$. Fix a complex $\mathcal{M}^\bullet$ in $\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. For any $\mathcal{U}_\lambda \in \mathcal{B}$ and for any factorization $f_\lambda = h_\lambda i_\lambda$, set $f_\lambda^\sharp\mathcal{M}^\bullet := (i_\lambda|h_\lambda)^\sharp\mathcal{M}^\bullet$. By 8.3.3, $f_\lambda^\sharp\mathcal{M}^\bullet$ does not depend on the choice of i_λ, h_λ. For

any $\mathcal{U}_\lambda \in \mathcal{B}$, any open subset $\mathcal{U}'$ of $\mathcal{U}_\lambda$ is also in $\mathcal{B}$ and by 8.4.1, with $\mathcal{U}_{\lambda'} := \mathcal{U}'$ we have $(f^\sharp_\lambda \mathcal{M}^\bullet)\big|_{\mathcal{U}_{\lambda'}} = (f^\sharp_{\lambda'}\mathcal{M}^\bullet)$. It follows that the differentials of $(f^\sharp_\lambda \mathcal{M}^\bullet)$ for $\lambda \in \Lambda$ can be pasted together to yield a differential on $f^\natural\mathcal{M}^\bullet$. We take this *canonical* choice as the differential of $f^\sharp\mathcal{M}^\bullet$.

It follows that our definition of $f^\sharp$ satisfies properties (ii) and (iii) of the Main Theorem in §1.3. We now address property (i) which amounts to verifying that the graded constructs of $(-)^\natural$ work at the level of complexes too.

Let us verify that for any map $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ in $\mathbb{F}_{\mathrm{c}}$, $f^\sharp$ is a functor of complexes, i.e., for any map of complexes $\mathcal{M}^\bullet \to \mathcal{N}^\bullet$ in $\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$, the canonical graded map $f^\sharp\mathcal{M}^\bullet \to f^\sharp\mathcal{N}^\bullet$ is also a map of complexes. The verification is local in nature and so we reduce to verifying at open sets of the type $\mathcal{U}_\lambda$ in $\mathcal{B}$; here the desired result is obvious.

Under the hypothesis of (i)(b) (resp. (i)(c)) of the Theorem, we set $C^\sharp_{f,g} = C^\natural_{f,g}$ (resp. $\delta^\sharp_{\mathcal{X}} = \delta^\natural_{\mathcal{X}}$). From 8.2.1 it follows that for any $(\mathcal{X}, \Delta) \in \mathbb{F}$ and $\mathcal{M}^\bullet \in \mathrm{Coz}_\Delta(\mathcal{X})$, the natural transformation $\delta^\sharp_{\mathcal{X}}$ induces a map of complexes $1^\sharp_{\mathcal{X}}\mathcal{M}^\bullet \xrightarrow{\sim} \mathcal{M}^\bullet$. The rest of this subsection is devoted to the case of the comparison maps.

Let

$$(\mathcal{X}, \Delta_{\mathcal{X}}) \xrightarrow{f} (\mathcal{Y}, \Delta_{\mathcal{Y}}) \xrightarrow{g} (\mathcal{Z}, \Delta_{\mathcal{Z}})$$

be maps in $\mathbb{F}_{\mathrm{c}}$ and let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}_{\Delta_{\mathcal{Z}}}(\mathcal{Z})$. We need to verify that the canonical graded isomorphism

$$C^\natural_{f,g}(\mathcal{M}^\bullet)\colon f^\sharp g^\sharp \mathcal{M}^\bullet \longrightarrow (gf)^\sharp\mathcal{M}^\bullet$$

is a map of complexes. Since the definition of $(-)^\natural$ is local in nature and since the definition of the differential for $(-)^\sharp$ is based on local constructions, we may assume without loss of generality that $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are affine formal schemes, say $\mathcal{X} = \mathrm{Spf}(C), \mathcal{Y} = \mathrm{Spf}(B)$ and $\mathcal{Z} = \mathrm{Spf}(A)$. By 2.4.3 we may factor the natural map $A \to B$ as

$$A \to P = ((A[X_1, \ldots, X_n])_S, I)^\wedge \overset{\pi}{\twoheadrightarrow} B$$

and the natural map $B \to C$ as

$$B \to Q = ((B[Y_1, \ldots, Y_m])_T, J)^\wedge \twoheadrightarrow C.$$

Set $\mathfrak{p} = \ker \pi$. Let $T' := \pi_Y^{-1}T$ where $\pi_Y\colon P[Y_1, \ldots, Y_m] \twoheadrightarrow B[Y_1, \ldots, Y_m]$ is the map naturally induced by π. Let J' be the inverse image of J under the induced surjection $(P[Y_1, \ldots, Y_m])_{T'} \twoheadrightarrow (B[Y_1, \ldots, Y_m])_T$; the kernel of this surjection is also generated by $\mathfrak{p}$. Set $R := ((P[Y_1, \ldots, Y_m])_{T'}, J')^\wedge$. Then

$$Q = ((P/\mathfrak{p}[Y_1, \ldots, Y_m])_T, J)^\wedge \cong ((P[Y_1, \ldots, Y_m])_{T'}/(\mathfrak{p}), J)^\wedge \cong R/(\mathfrak{p}R).$$

Thus the following statements hold:

- As a map of noetherian adic rings the natural map $P \to R$ is essentially of pseudo-finite type and formally smooth.
- $\mathrm{Spf}(Q)$ is the fibered product $\mathrm{Spf}(B) \times_{\mathrm{Spf}(P)} \mathrm{Spf}(R)$ in $\mathbb{F}$.

Now consider the following diagram of canonical maps where the vertical ones are smooth and the horizontal ones are closed immersions.

$$\begin{tikzcd}
\mathrm{Spf}(C) \arrow[r, "i_2"] & \mathrm{Spf}(Q) \arrow[r, "i'_1"] \arrow[d, "h'_2"'] & \mathrm{Spf}(R) \arrow[d, "h_2"'] \\
& \mathrm{Spf}(B) \arrow[r, "i_1"] & \mathrm{Spf}(P) \arrow[d, "h_1"'] \\
& & \mathrm{Spf}(A)
\end{tikzcd}$$

Note that $f = h'_2 i_2$, $g = h_1 i_1$. By construction, we have

$$f^\sharp = (i_2|h'_2)^\sharp, \quad g^\sharp = (i_1|h_1)^\sharp, \quad (gf)^\sharp = (i'_1 i_2|h_1 h_2)^\sharp.$$

To prove that $C^\natural_{f,g}$ is a map of complexes consider the following commutative diagram of canonical graded isomorphisms.

$$\begin{tikzcd}
(h'_2 i_2)^\natural (h_1 i_1)^\natural \arrow[rr, "C^\natural_{f,g}"] \arrow[d] & & (h_1 h_2 i'_1 i_2)^\natural \arrow[d] \\
i_2^\natural h_2'^\natural i_1^\natural h_1^\natural \arrow[r, "\alpha"] & i_2^\natural i_1'^\natural h_2^\natural h_1^\natural \arrow[r, "\beta"] & (i'_1 i_2)^\natural (h_1 h_2)^\natural
\end{tikzcd}$$

The bottom row, when equipped with differentials as in §8.1 and §8.2, is seen to be a map of complexes by using Case 3 of §8.3 for α and Cases 1, 2 for β. Therefore $C^\natural_{f,g}$ is a map of complexes.

8.5. Compatibility with translations. For any map $f\colon (\mathcal{X}, \Delta) \to (\mathcal{Y}, \Delta')$ in $\mathbb{F}_c$ and for any integer n, we denote the obvious map $(\mathcal{X}, \Delta - n) \to (\mathcal{Y}, \Delta' - n)$ by $f^{(n)}$. Let $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta'}(\mathcal{Y})$. Then $\mathcal{M}^\bullet[n] \in \mathrm{Coz}_{\Delta'-n}(\mathcal{Y})$ and both $(f^\sharp \mathcal{M}^\bullet)[n]$ and ${f^{(n)}}^\sharp(\mathcal{M}^\bullet[n])$ are in $\mathrm{Coz}_{\Delta-n}(\mathcal{X})$. The latter two also have the same underlying graded objects and so we are naturally led to comparing their differentials.

Consider the graded isomorphism

$$ {f^{(n)}}^\sharp(\mathcal{M}^\bullet[n]) \xrightarrow{\ \sim\ } (f^\sharp \mathcal{M}^\bullet)[n] \tag{85}$$

defined punctually, say at a point $x \in \mathcal{X}$, with $y = f(x)$, by

$$ ({f^{(n)}}^\sharp(\mathcal{M}^\bullet[n]))(x) = \widehat{f_x}^\sharp(\mathcal{M}^\bullet(y)) \xrightarrow{(-1)^{nt}} \widehat{f_x}^\sharp(\mathcal{M}^\bullet(y)) = ((f^\sharp \mathcal{M}^\bullet)[n])(x), \tag{86}$$

where t is the transcendence degree of the residue field extension $k(y) \to k(x)$.

Proposition 8.5.1. *The graded isomorphism in* (85) *is an isomorphism of complexes. Morever the following hold.*

(i) *If m is an integer, then the following diagram of isomorphisms commutes, where the horizontal arrows are obtained using* (85).

$$\begin{tikzcd}
{f^{(m+n)}}^\sharp(\mathcal{M}^\bullet[m+n]) \arrow[rr] \arrow[d, equal] & & (f^\sharp \mathcal{M}^\bullet)[m+n] \arrow[d, equal] \\
{f^{(m)(n)}}^\sharp(\mathcal{M}^\bullet[m][n]) \arrow[r] & ({f^{(m)}}^\sharp(\mathcal{M}^\bullet[m]))[n] \arrow[r] & (f^\sharp \mathcal{M}^\bullet)[m][n]
\end{tikzcd}$$

(ii) *If* $g\colon (\mathcal{Y},\Delta') \to (\mathcal{Z},\Delta'')$ *is a map in* $\mathbb{F}_c$ *then the following diagram of isomorphisms commutes, where the horizontal arrows are obtained using* (85).

$$\begin{tikzcd}
f^{(n)\sharp}g^{(n)\sharp}(\mathcal{M}^\bullet[n]) \arrow[r] \arrow[d, "\text{via } C^\sharp_{f^{(n)},g^{(n)}}"'] & f^{(n)\sharp}((g^\sharp\mathcal{M}^\bullet)[n]) \arrow[r] & (f^\sharp g^\sharp\mathcal{M}^\bullet)[n] \arrow[d, "\text{via } C^\sharp_{f,g}"] \\
(g^{(n)}f^{(n)})^\sharp(\mathcal{M}^\bullet[n]) \arrow[r, equal] & (gf)^{(n)\sharp}(\mathcal{M}^\bullet[n]) \arrow[r] & ((gf)^\sharp\mathcal{M}^\bullet)[n]
\end{tikzcd}$$

Proof. The horizontal maps in (i) and (ii) are, at the punctual level, signed multiples of identity maps, according to the sign $(-1)^{nt}$ in (86). Hence (i) follows from the additivity of nt vis-à-vis n for fixed t, while (ii) follows from the additivity vis-à-vis t for fixed n.

To verify that (85) is a map of complexes it suffices to do so locally on $\mathcal{X}$ and $\mathcal{Y}$ and hence we may assume that f factors as a closed immersion into a smooth map. Thus it suffices to assume that f is either a closed immersion or a smooth map. The first case is easily settled, using, e.g., (50). (In particular, sign considerations play no role.) Assume then that f is a smooth map.

We use notation as in §8.1. From our convention in §1.4(iv), we see that

$$f^*\mathcal{M}^\bullet[n]\otimes_{\mathcal{X}}\omega_f[d] = (f^*\mathcal{M}^\bullet\otimes_{\mathcal{X}}\omega_f[d])[n].$$

Recall that for any complex $\mathcal{F}^\bullet$, $E_{\Delta-n}(\mathcal{F}^\bullet[n]) = (E_\Delta\mathcal{F}^\bullet)[n]$ (see (11)). It follows that

$$\mathbb{E}_{f^{(n)}}(\mathcal{M}^\bullet[n]) = (\mathbb{E}_f\mathcal{M}^\bullet)[n].$$

To conclude the smooth case, it suffices to show that the preceding equality is consistent with the isomorphism of (85), i.e., for any $x\in\mathcal{X}$, the following diagram of punctual isomorphisms commutes.

$$\begin{tikzcd}
(\mathbb{E}_{f^{(n)}}(\mathcal{M}^\bullet[n]))(x) \arrow[r, equal] \arrow[d, "\text{via (76)}"'] & ((\mathbb{E}_f\mathcal{M}^\bullet)[n])(x) \arrow[d, "\text{via (76)}"] \\
\widehat{f_x}^\sharp(\mathcal{M}^\bullet(y)) \arrow[r, "(-1)^{nt}"] & \widehat{f_x}^\sharp(\mathcal{M}^\bullet(y))
\end{tikzcd}$$

Using $p=\Delta(x), q=\Delta'(y)$ as in §8.1, we have that $t=p-q$. After some diagram chase, using the choices of signs involved, one reduces the commutativity to the following immediately verifiable calculation.

$$(-1)^{(p-n)(q-n)+p-n}(-1)^{n(p-q)}(-1)^{pq+p} = 1$$

$\square$

9. Residual and dualizing complexes

In this section we discuss residual complexes on a formal scheme $\mathcal{X}$ (Definition 9.1.1.)[6] These special Cousin complexes play an important role in duality theory. They can be characterized as being the Cousin complexes associated to *pointwise dualizing* complexes in the derived category $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ (Proposition 9.2.2). For any $\mathbb{F}_{\mathrm{c}}$-map f, $f^\sharp$ takes residual complexes to residual complexes (Proposition 9.1.4).

On finite-type schemes over finite-dimensional regular noetherian rings, Yekutieli and Zhang have found a duality between the category of coherent sheaves and the category of Cohen-Macaulay complexes with coherent homology [**28**, Thm. 8.9]. This duality generalizes to all formal schemes having a coherent-dualizing complex (Corollary 9.3.2). (For the formal spectrum of a complete local ring it is essentially Matlis duality, see Example 9.3.4.) There follow quick proofs of a number of facts about Cousin complexes, for example a generalization to complexes on formal schemes, of results of Dibaei, Tousi and Kawasaki about finite generation of the homology of Cousin complexes of finite type modules (see Proposition 9.3.5).

9.1. $(-)^\sharp$ preserves residual.

Definition 9.1.1. We say that a complex $\mathcal{R}^\bullet$ on a noetherian formal scheme $\mathcal{X}$ is a residual complex if it consists of $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$-modules and if for any defining ideal $\mathcal{J}$, the complex $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}/\mathcal{J}, \mathcal{R}^\bullet)$ on the scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{J})$ is a residual complex.

It is easily shown that $\mathcal{R}^\bullet$ is residual if there exists one defining ideal $\mathcal{J}$ such that for any $n > 0$, $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}/\mathcal{J}^n, \mathcal{R}^\bullet)$ is residual on $(\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{J}^n)$. Note that the existence of a residual complex on a noetherian formal scheme $\mathcal{X}$ implies that $\mathcal{X}$ is in $\mathbb{F}$ because the corresponding statement for ordinary schemes is true.

Lemma 9.1.2 (cf. [**11**, VI, Lemma 5.2]). *Let A be a noetherian local ring with maximal ideal m_A and residue field k. Let M be an m_A-torsion A-module. Suppose there exists an ideal I in A such that for any integer $n > 0$, $\mathrm{Hom}_A(A/I^n, M)$ is an injective hull of k over A/I^n. Then M is an injective hull of k over A.*

Proof. Set $M_n := \mathrm{Hom}_A(A/I^n, M) \subset M$. Each M_n is an essential extension of k and $M = \cup_n M_n$. Therefore M is also an essential extension of k. In particular, there exists an embedding $M \subset E(k)$ where $E(k)$ is an injective hull of k over A. Set $E_n := \mathrm{Hom}_A(A/I^n, E(k)) \subset E(k)$. By hypothesis, for each n, $M_n = E_n$. Therefore, $M = E(k)$. □

For the next result we use some basic facts on residual complexes on ordinary schemes as developed in [**11**, VI, §1] and [**5**, §3.1, §3.2].

Lemma 9.1.3. *Let $\mathcal{X}$ be a noetherian formal scheme and $\mathcal{R}^\bullet$ a residual complex on $\mathcal{X}$. Then there is a unique codimension function Δ on $\mathcal{X}$ such that $\mathcal{R}^\bullet \in \mathrm{Coz}_\Delta(\mathcal{X})$. Moreover, for any $x \in \mathcal{X}$, $\mathcal{R}^\bullet(x)$ is an $\mathcal{O}_{\mathcal{X},x}$-injective hull of the residue field $k(x)$ at x.*

Proof. Let $\mathcal{I}$ be a defining ideal in $\mathcal{O}_\mathcal{X}$. For $n > 0$, set

$$\mathcal{R}^\bullet_n := \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}/\mathcal{I}^n, \mathcal{R}^\bullet).$$

Since $\mathcal{R}^\bullet$ consists of $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$-modules, it is isomorphic to the direct limit of the $\mathcal{R}^\bullet_n$'s. On the ordinary scheme $X_n := (\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{I}^n)$, the complex $\mathcal{R}^\bullet_n$, being a residual

[6] This definition agrees with the one for ordinary schemes in [**11**, VI, §1], and also, for formal schemes of finite Krull dimension, with the one in [**27**, 5.9]. (Yekutieli's residual complexes are dualizing, hence cannot exist on formal schemes of infinite Krull dimension [**11**, p. 283, Cor. 7.2].)

complex, induces a unique codimension function Δ_n for which $\mathcal{R}_n^\bullet \in \mathrm{Coz}_{\Delta_n}(X_n)$. Moreover, for every $x \in \mathcal{X}$, $\mathcal{R}_n^\bullet(x)$ is an $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_x^n$-injective hull of $k(x)$. Since $\mathcal{R}_n^\bullet \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}/\mathcal{I}^n, \mathcal{R}_{n+1}^\bullet)$, $\mathcal{R}_n^\bullet$ is also in $\mathrm{Coz}_{\Delta_{n+1}}(X_n)$ (cf. 6.1.1).In particular, $\Delta_n = \Delta_{n+1}$, which we henceforth denote by Δ. Moreover, the canonical inclusion $\mathcal{R}_n^\bullet \to \mathcal{R}_{n+1}^\bullet$ respects the pointwise decomposition. Taking limits we see that $\mathcal{R}^\bullet$ is a Δ-Cousin complex, where for any x, $\mathcal{R}^\bullet(x)$ is a direct limit of the $\mathcal{R}_n^\bullet(x)$'s. Since each $\mathcal{R}_n^\bullet(x)$ is an m_x-torsion $\mathcal{O}_{\mathcal{X},x}$-module, so is $\mathcal{R}^\bullet(x)$. By 9.1.2, $\mathcal{R}^\bullet(x)$ is an injective hull of $k(x)$ over $\mathcal{O}_{\mathcal{X},x}$. □

In light of 9.1.3, if a codimension function Δ is chosen on $\mathcal{X}$, then we shall only consider those residual complexes which induce Δ. Thus, for $(\mathcal{X}, \Delta) \in \mathbb{F}_\mathrm{c}$, a residual complex shall always be assumed, by default, to lie in $\mathrm{Coz}_\Delta(\mathcal{X})$.

Proposition 9.1.4. *Let $f\colon (\mathcal{X}, \Delta_\mathcal{X}) \to (\mathcal{Y}, \Delta_\mathcal{Y})$ be a map in $\mathbb{F}_\mathrm{c}$. If $\mathcal{R}^\bullet$ is a residual complex on $\mathcal{Y}$, then $f^\sharp\mathcal{R}^\bullet$ is a residual complex on $\mathcal{X}$.*

Proof. First we prove the result when $X := \mathcal{X}$ and $Y := \mathcal{Y}$ are ordinary schemes so that f is essentially of finite type. For any $x \in X$, $(f^\sharp\mathcal{R}^\bullet)(x)$ is an $\mathcal{O}_{X,x}$-injective hull of $k(x)$ at x (4.3.1, IV). So it only remains to check that $f^\sharp\mathcal{R}^\bullet$ has coherent homology. This being a local property, in view of the existence of factorizations as in 2.4.4, it suffices to consider the cases when f is a smooth map (of constant relative dimension n) and when f is a closed immersion. In the smooth situation, by Cohen-Macaulayness of $f^*\mathcal{R}^\bullet \otimes_X \omega_f[n]$ (5.1.3), there are isomorphisms $f^*\mathcal{R}^\bullet \otimes_X \omega_f[n] \xrightarrow{\sim} E(f^*\mathcal{R}^\bullet \otimes_X \omega_f[n]) \xrightarrow{\sim} f^\sharp\mathcal{R}^\bullet$ (see 3.3.2 and (76), though a-priori, the first isomorphism is only a $\mathbf{D}(\mathcal{X})$-isomorphism). Since ω_f is coherent, $f^\sharp\mathcal{R}^\bullet$ has coherent homology. If f is a closed immersion, in view of the isomorphism $f^\sharp \xrightarrow{\sim} f^\flat$ of (78), the proof follows easily from the coherence of $f_*\mathcal{O}_\mathcal{X}$ (cf. [**11**, III, Prop. 6.1]).

Now we consider the general case, when $\mathcal{X}$, $\mathcal{Y}$ are arbitrary formal schemes. Fix a defining ideal $\mathcal{I}$ in $\mathcal{O}_\mathcal{X}$ and a defining ideal $\mathcal{J}$ in $\mathcal{O}_\mathcal{Y}$ such that $\mathcal{J}\mathcal{O}_\mathcal{X} \subset \mathcal{I}$. Set $X_n := (\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{I}^n), Y_n := (\mathcal{Y}, \mathcal{O}_\mathcal{Y}/\mathcal{J}^n)$. Then there is a commutative diagram of natural induced maps as follows.

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle i_n} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

Using the isomorphisms $i_n^\sharp \xrightarrow{\sim} i_n^\flat$ and $j_n^\sharp \xrightarrow{\sim} j_n^\flat$ obtained from (78), we deduce that $\mathcal{R}^\bullet$ is residual on $\mathcal{Y} \iff \forall n$, $i_n^\sharp\mathcal{R}^\bullet$ is residual on $Y_n \Longrightarrow \forall n$, $f_n^\sharp i_n^\sharp\mathcal{R}^\bullet$ is residual on $X_n \iff \forall n$, $j_n^\sharp f^\sharp\mathcal{R}^\bullet$ is residual on $X_n \Longrightarrow f^\sharp\mathcal{R}^\bullet$ is residual on $\mathcal{X}$. □

9.2. Residual and pointwise dualizing complexes. This subsection is devoted to showing a pointwise dualizing property for residual complexes. Let $\mathcal{X}$ be a noetherian formal scheme in $\mathbb{F}$. For any point $x \in \mathcal{X}$, let j_x denote the canonical map $\mathrm{Spf}(\widehat{\mathcal{O}_{\mathcal{X},x}}) \to \mathcal{X}$ where $\widehat{\mathcal{O}_{\mathcal{X},x}}$ is the completion of the local ring $\mathcal{O}_{\mathcal{X},x}$ along the stalk $\mathcal{I}_x$ of a defining ideal $\mathcal{I}$ in $\mathcal{O}_\mathcal{X}$.

Lemma 9.2.1. *Let $\mathcal{X}, x, j := j_x$ be as above. Set $\mathcal{W} := \mathrm{Spf}(\widehat{\mathcal{O}_{\mathcal{X},x}})$. Then:*

(i) *The map j is adic, i.e., for any defining ideal $\mathcal{I}$ in $\mathcal{O}_\mathcal{X}$, $\mathcal{I}\mathcal{O}_\mathcal{W}$ is a defining ideal. In particular, j^* takes $\mathcal{A}_\mathrm{t}(\mathcal{X})$ to $\mathcal{A}_\mathrm{t}(\mathcal{W})$ and $\mathcal{A}_\mathrm{qct}(\mathcal{X})$ to $\mathcal{A}_\mathrm{qct}(\mathcal{W})$. Moreover, for any $\mathcal{F} \in \mathcal{A}_\mathrm{t}(\mathcal{X})$, the natural map $j^{-1}\mathcal{F} \to j^*\mathcal{F}$ is an isomorphism.*

(ii) *The map j is étale (smooth of relative dimension 0). In particular, j^* is exact and $\mathbf{L}j^* = j^*$ takes $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$ to $\mathbf{D}_{\mathrm{t}}(\mathcal{W})$ and $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ to $\mathbf{D}_{\mathrm{qct}}(\mathcal{W})$.*

(iii) *If w denotes the unique closed point of $\mathcal{W}$, then for any $\mathcal{F}^\bullet \in \mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$, there is a natural isomorphism $H^i_w j^*\mathcal{F}^\bullet \xrightarrow{\sim} H^i_x\mathcal{F}^\bullet$.*

Proof. First we show that j is adic and étale. Let $\mathcal{V} = \mathrm{Spf}(A)$ be an affine neighborhood of x. It suffices to show that the induced map $\mathcal{W} \to \mathcal{V}$ is adic and étale, so we assume $\mathcal{V} = \mathcal{X}$. Let I be a defining ideal of A. Then $\mathcal{I} = I^{\sim_A}$ is a defining ideal in $\mathcal{O}_{\mathcal{X}}$ and since $I\mathcal{O}_{\mathcal{X},x} = \mathcal{I}_x$ therefore I generates a defining ideal of $\widehat{\mathcal{O}_{\mathcal{X},x}}$. By 2.2.4 we see that the image of $j^*\mathcal{I}$ in $\mathcal{O}_{\mathcal{W}}$ is the ideal generated by $I\widehat{\mathcal{O}_{\mathcal{X},x}}$. Thus j is adic. For étaleness we refer to the discussion preceding 2.6.9.

For an adic map, $(-)^*$ sends $\mathcal{A}_{\mathrm{t}}$-modules to $\mathcal{A}_{\mathrm{t}}$-modules, as can be seen by arguing at the stalks using 3.1.1. Since $(-)^*$ also preserves quasi-coherence, it sends $\mathcal{A}_{\mathrm{qct}}$-modules to $\mathcal{A}_{\mathrm{qct}}$-ones. Exactness of j^* follows from flatness of j which in turn follows from smoothness of j. Finally, since j^{-1} and j^* commute with direct limits, to verify that $j^{-1}\mathcal{F} \xrightarrow{\sim} j^*\mathcal{F}$ for $\mathcal{F} \in \mathcal{A}_{\mathrm{t}}(\mathcal{X})$, it suffices to do so when $\mathcal{F}$ is annihilated by a defining ideal $\mathcal{I}$ in $\mathcal{O}_{\mathcal{X}}$. Now we may descend to the corresponding schemes and here the result is clear. Thus (i) and (ii) are proved.

For (iii), by 3.1.8(ii), we may assume that $\mathcal{F}^\bullet$ consists of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injectives. Then with m_x, m_w as the local rings at x, w respectively, there are natural isomorphisms $H^i_{m_x}\mathcal{F}^\bullet_x \xrightarrow{\sim} H^i_x\mathcal{F}^\bullet$ and $H^i_{m_w}(j^*\mathcal{F}^\bullet)_w \xrightarrow{\sim} H^i_w j^*\mathcal{F}^\bullet$ (see (6) of §3.1). Now (iii) follows by using $(j^*\mathcal{F}^\bullet)_w = (j^{-1}\mathcal{F}^\bullet)_w = \mathcal{F}^\bullet_x$, and that m_w is just the completion of m_x. $\square$

For the definition of a *t-dualizing* (= *torsion-dualizing*) complex on a noetherian formal scheme $\mathcal{X}$ we refer to [**2**, 2.5.1]. (See also [**27**, §5].) An $\mathcal{O}_{\mathcal{X}}$-complex $\mathcal{D}^\bullet$ is *pointwise t-dualizing* if $\mathcal{D}^\bullet \in \mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$ and for any $x \in \mathcal{X}$, with $j_x, \mathcal{W}$ as in 9.2.1, $j_x^*\mathcal{D}^\bullet$ is t-dualizing on $\mathcal{W}$. In view of [**5**, 3.1.4], this definition agrees with the one for ordinary schemes .

We need some notation before proceeding further. Let $\mathcal{Y}$ be a noetherian formal scheme, and $\mathcal{I}$ a defining ideal in $\mathcal{O}_{\mathcal{Y}}$. Let i be the canonical closed immersion $Y := (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\mathcal{I}) \to \mathcal{Y}$. Then for any $\mathcal{F}^\bullet \in \mathbf{D}(\mathcal{Y})$ we set

$$i^!\mathcal{F}^\bullet := \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}}/\mathcal{I}, \mathcal{F}^\bullet) \in \mathbf{D}(Y).$$

Here $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}/\mathcal{I}, \mathcal{F}^\bullet)$ can be considered as a complex in $\mathbf{D}(Y)$ in a natural way. Also, $i^!$ is a right adjoint of $i_* = \mathbf{R}i_* \colon \mathbf{D}_{\mathrm{qc}}(Y) \to \mathbf{D}(\mathcal{Y})$ (see [**2**, Examples 6.1.3(4)]).

Proposition 9.2.2. *Let $\mathcal{X}$ be a noetherian formal scheme.*

(i) *Let $\mathcal{D}^\bullet$ be a complex on $\mathcal{X}$ and let $\mathcal{I}$ be a defining ideal in $\mathcal{O}_{\mathcal{X}}$. For $n > 0$, let i_n denote the canonical immersion $X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^n) \to \mathcal{X}$. Then $\mathcal{D}^\bullet$ is pointwise t-dualizing $\iff$ for any $n > 0$, $i_n^!\mathcal{D}^\bullet$ is pointwise dualizing on X_n.*

(ii) *Let $\mathcal{D}^\bullet$ be a complex on $\mathcal{X}$. Then $\mathcal{D}^\bullet$ is t-dualizing $\iff$ $\mathcal{X}$ has finite Krull dimension and $\mathcal{D}^\bullet$ is pointwise t-dualizing.*

(iii) *Any residual complex on $\mathcal{X}$ is pointwise t-dualizing. Conversely, if $\mathcal{D}^\bullet$ is a pointwise t-dualizing complex on $\mathcal{X}$, then the following hold:*

(a) *The complex $\mathcal{D}^\bullet$ induces a codimension function Δ on $\mathcal{X}$ such that $\mathcal{D}^\bullet$ is Cohen-Macaulay w.r.t. Δ.*

(b) *The complex $E_\Delta\mathcal{D}^\bullet$ is residual, so that, via $1 \cong QE_\Delta$ of 3.3.2, $\mathcal{D}^\bullet$ is isomorphic to a residual complex.*

PROOF. (i) The flat-base-change isomorphism [**2**, p. 8, Thm. 3] gives:

LEMMA 9.2.3. *Let $\mathcal{X}, x, j_x, \mathcal{W}$ be as in 9.2.1. Let $\mathcal{I}$ be a defining ideal in $\mathcal{O}_{\mathcal{X}}$. Let X, W be the schemes obtained by going modulo $\mathcal{I}, \mathcal{I}\mathcal{O}_{\mathcal{W}}$ in $\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{W}}$ respectively so that the following diagram of induced natural maps is a fiber square.*

$$\begin{tikzcd} W \arrow[r, "j'_x"] \arrow[d, "i'"'] & X \arrow[d, "i"] \\ \mathcal{W} \arrow[r, "j_x"'] & \mathcal{X} \end{tikzcd}$$

Then for any $\mathcal{F}^\bullet \in \mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$ there is a natural isomorphism $i'^! j_x^* \mathcal{F}^\bullet \xrightarrow{\sim} j'^*_x i^! \mathcal{F}^\bullet$.

Now, in 9.2.3 take $i = i_n, X = X_n$. Then $\mathcal{D}^\bullet$ is pointwise t-dualizing $\Longrightarrow$ (for any $x \in \mathcal{X}$) $j_x^*\mathcal{D}^\bullet$ is t-dualizing on $\mathcal{W} \Longrightarrow i'^! j_x^* \mathcal{D}^\bullet$ is dualizing on W ([**2**, Lemma 2.5.10]) $\Longrightarrow j'^*_x i^! \mathcal{D}^\bullet$ is dualizing on W. Thus $i^!\mathcal{D}^\bullet$ is a pointwise dualizing complex on X. The converse is proved by reversing the arguments.

(ii). We use the same arguments as in (i) above along with [**11**, V, 7.2 and 8.2].

(iii). Suppose $\mathcal{R}^\bullet$ is a residual complex on $\mathcal{X}$. First note that if $\mathcal{X}$ has finite Krull dimension, then $\mathcal{R}^\bullet$ is t-dualizing. Indeed, by 9.1.3 and 2.3.6(ii), $\mathcal{R}^\bullet$ consists of $\mathcal{A}_{\mathrm{qct}}$-injectives, therefore, by [**2**, 2.5.6], [**2**, 2.5.10] and [**11**, VI, 1.1(a)], $\mathcal{R}^\bullet$ is t-dualizing. Now we drop the hypothesis on Krull dimension of $\mathcal{X}$. For $x \in \mathcal{X}$, j_x in 9.2.1 is étale and adic. Therefore, by (76), we obtain an isomorphism $j_x^*\mathcal{R}^\bullet \xrightarrow{\sim} j_x^\sharp \mathcal{R}^\bullet$, which by 9.1.4, is a residual complex on $\mathcal{W}$. Since $\mathcal{W}$ has finite Krull dimension, $j_x^*\mathcal{R}^\bullet$ is t-dualizing on $\mathcal{W}$. Thus $\mathcal{R}^\bullet$ is a pointwise t-dualizing complex.

Conversely, suppose $\mathcal{D}^\bullet$ is a pointwise t-dualizing complex on $\mathcal{X}$. By 3.1.8(ii) we may assume that $\mathcal{D}^\bullet$ is a bounded below complex of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injectives. For a defining ideal $\mathcal{I}$ in $\mathcal{O}_{\mathcal{X}}$ and $n > 0$, set $\mathcal{D}^\bullet_n := \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n, \mathcal{D}^\bullet)$. Then $\mathcal{D}^\bullet$ is isomorphic to the direct limit of the $\mathcal{D}^\bullet_n$'s. By [**2**, 2.5.6] $\mathcal{D}^\bullet_n \xrightarrow{\sim} i_n^! \mathcal{D}^\bullet$ and hence by [**2**, 2.5.10] $\mathcal{D}^\bullet_n$ is a pointwise dualizing complex on $X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^n)$. In particular, $\mathcal{D}^\bullet_n$ induces a unique function Δ_n for which it is Cohen-Macaulay. Since $\mathcal{D}^\bullet_n \xrightarrow{\sim} i^!_{n,n+1}\mathcal{D}^\bullet_{n+1}$ where $i_{n,n+1}$ is the canonical immersion $X_n \to X_{n+1}$, it follows that $\Delta_n = \Delta_{n+1}$, henceforth to be denoted as Δ. Since $H^*_x(-)$ commutes with direct limits we see that the Δ-Cohen-Macaulayness of the $\mathcal{D}^\bullet_n$'s carries over to $\mathcal{D}^\bullet$. By 2.3.6(iii),(iv) and 3.1.3, there is an isomorphism $H^*_x\mathcal{D}^\bullet \xrightarrow{\sim} H^*_{m_x}\mathcal{D}^\bullet_x$ so that $E_\Delta\mathcal{D}^\bullet \in \mathrm{Coz}_\Delta(\mathcal{X})$. Thus, via the isomorphism in 3.3.2, $\mathcal{D}^\bullet$ is $\mathbf{D}(X)$-isomorphic to a complex $\mathcal{R}^\bullet$ in $\mathrm{Coz}_\Delta(\mathcal{X})$.

If we define $\mathcal{R}^\bullet_n$ in the obvious manner, then $\mathcal{R}^\bullet_n$ is a pointwise dualizing complex on X_n lying in $\mathrm{Coz}_\Delta(X_n)$. In particular, $\mathcal{R}^\bullet_n$ is a residual complex on X_n. Thus $\mathcal{R}^\bullet$ is a residual complex. □

Finally we prove that Yekutieli's definition of a residual complex on $\mathcal{X}$ agrees with ours when $\mathcal{X}$ has finite Krull dimension,[7] i.e., we show that a complex $\mathcal{R}^\bullet$ on $\mathcal{X}$ is residual if and only if it is t-dualizing and there is an isomorphism of $\mathcal{O}_{\mathcal{X}}$-modules $\oplus_p \mathcal{R}^p \xrightarrow{\sim} \oplus_x i_x J(x)$ where $J(x)$ is the $\mathcal{O}_{\mathcal{X},x}$-injective hull of the residue field $k(x)$ at x (see 2.3.6(ii)). The 'only if' part follows from 9.1.3 and 9.2.2(ii),(iii). For the 'if' part, note that the corresponding statement holds for ordinary schemes and hence if we define $\mathcal{R}^\bullet_n$ as in the proof of 9.1.3 then $\mathcal{R}^\bullet_n$ is residual on X_n. Thus $\mathcal{R}^\bullet$ is residual.

[7]In this context note that Lemma 5.13 in [**27**] requires a finite-dimensionality *assumption*, see Nagata's example of an infinite-dimensional noetherian regular ring in [**21**, p. 203].

9.3. CM complexes and coherent sheaves. For a formal scheme $\mathscr{X}$, let $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$ be the essential image of $\mathbf{R}\varGamma'_{\mathscr{X}}|_{\mathbf{D}_{\mathrm{c}}}$, i.e., the full subcategory of $\mathbf{D}(\mathscr{X})$ such that $\mathcal{E} \in \mathbf{D}_{\mathrm{c}}^* \Leftrightarrow \mathcal{E} \cong \mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{F}$ with $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}$. Proposition 3.1.8 shows that $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X}) \subset \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$. The functor $\mathbf{R}\varGamma'_{\mathscr{X}}$ is an equivalence from the category $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ to the category $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$, with quasi-inverse $\mathbf{\Lambda} := \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}},\, -)$ (see [**2**, p. 24]). In particular, if $\mathscr{X}$ is an ordinary scheme, so that $\mathbf{R}\varGamma'_{\mathscr{X}}$ is the identity functor on $\mathbf{D}(\mathscr{X})$, then $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X}) = \mathbf{D}_{\mathrm{c}}(\mathscr{X})$.

By [**2**, p. 26, 2.5.3 and p. 27, 2.5.5], $\mathscr{X}$ has a t-dualizing complex $\mathcal{R}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$ iff $\mathscr{X}$ has a "c-dualizing" complex $\mathcal{R}_{\mathrm{c}}^\bullet \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$, which is so, e.g., if $\mathscr{X}$ is locally embeddable in a regular finite-dimensional formal scheme. In fact $\mathbf{\Lambda}\mathcal{R}^\bullet$ is c-dualizing; and conversely, if $\mathcal{R}_{\mathrm{c}}^\bullet$ is c-dualizing then $\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{R}_{\mathrm{c}}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$ is t-dualizing.

Proposition 9.3.1. *Let $\mathscr{X}$ be a formal scheme with a t-dualizing complex $\mathcal{R}^\bullet$, which may be assumed residual, and Δ the codimension function associated to $\mathcal{R}^\bullet$ (see 9.2.2, 9.1.3). Assume further that $\mathcal{R}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$. Then $\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$ is a Δ-*CM *complex if and only if its dual*

$$\mathcal{D}_{\mathrm{t}}\mathcal{G}^\bullet := \mathbf{R}\mathcal{H}om^\bullet(\mathcal{G}^\bullet, \mathcal{R}^\bullet)$$

is ($\mathbf{D}(\mathscr{X})$-isomorphic to) a coherent $\mathcal{O}_{\mathscr{X}}$-module.

By [**2**, p. 28, 2.5.8], the functor $\mathcal{D}_{\mathrm{t}}$ induces, in either direction, an antiequivalence between $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ and $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$. Thus:

Corollary 9.3.2. *The functor $\mathcal{D}_{\mathrm{t}}$ induces, in either direction, an antiequivalence between $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ and the full subcategory of* Δ-CM *complexes in $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$.*

Since for all $\mathcal{M}^\bullet \in \mathbf{D}(\mathscr{X})$, $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathbf{\Lambda}\mathcal{R}^\bullet) \cong \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{M}^\bullet, \mathcal{R}^\bullet)$ (see [**2**, p. 24, (2.5.0.1)]), therefore:

Corollary 9.3.3. *A complex $\mathcal{M}^\bullet \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$ is such that $\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{M}^\bullet$ is* Δ-CM *if and only if, with $\mathcal{R}_{\mathrm{c}}^\bullet := \mathbf{\Lambda}\mathcal{R}^\bullet$, the coherent dual*

$$\mathcal{D}_{\mathrm{c}}\mathcal{M}^\bullet := \mathbf{R}\mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{R}_{\mathrm{c}}^\bullet)$$

is a coherent $\mathcal{O}_{\mathscr{X}}$-module.

In particular, $\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}$ is Δ-CM *if and only if $\mathcal{R}_{\mathrm{c}}^\bullet$ is a coherent $\mathcal{O}_{\mathscr{X}}$-module.*

Example 9.3.4. Let A be a complete local ring and $\mathscr{X} := \mathrm{Spf}(A)$ the formal spectrum of A. Then $\mathcal{A}(\mathscr{X})$ is the category of A-modules. We can take $\mathcal{R}^\bullet$ to be an injective hull of the residue field of A, (considered as a complex vanishing in nonzero degrees) [**2**, p. 25, Example 2.5.2(3)]. Then Δ maps the unique point $x \in \mathscr{X}$ to 0; and a Δ-Cousin complex is simply an A-module. Thus the Cousin functor E can be identified with the functor H_x^0, and by 3.3.1, H_x^0 is an *equivalence* from the category of Δ-CM complexes to the category of A-modules. So if $\mathcal{G}^\bullet$ is Δ-CM then there is a $\mathbf{D}$-isomorphism $\mathcal{G}^\bullet \xrightarrow{\sim} H_x^0\mathcal{G}^\bullet$, whence $\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X}) \Leftrightarrow H_x^0\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$. But from [**2**, p. 28, Prop. 2.5.8(a)] and Matlis duality [**19**, p. 148, Thm. 18.6(v)] it follows that $H_x^0\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$ if and only if $H_x^0\mathcal{G}^\bullet$ is an *artinian* A-module. Thus:

The equivalence of categories $H_x^0\colon \{\Delta\text{-CM } \mathcal{O}_{\mathscr{X}}\text{-complexes}\} \xrightarrow{\approx} \{A\text{-modules}\}$ takes the full subcategory of Δ-CM *complexes in $\mathbf{D}_{\mathrm{c}}^*(\mathscr{X})$ to the full subcategory of* artinian *A-modules, and transforms the antiequivalence of* 9.3.2 *into Matlis duality.*

Proof of 9.3.1. As above, if $\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{c}}^*$ then $\mathcal{D}_{\mathrm{t}}\mathcal{G}^\bullet \in \mathbf{D}_{\mathrm{c}}$, and $\mathcal{G}^\bullet \cong \mathcal{D}_{\mathrm{t}}\mathcal{D}_{\mathrm{t}}\mathcal{G}^\bullet$. It will therefore suffice to show that *if $x \in \mathscr{X}$ and $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{c}}$ then*

$$H^i\mathbf{R}\varGamma_{\!x}\,\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{R}^\bullet) = 0 \iff \mathrm{H}^{\Delta(x)-i}\mathcal{F}_x^\bullet = 0. \tag{9.3.1.1}$$

Let us prove (9.3.1.1). The first assertion in [**1**, p. 33, (5.2.1)] (whose proof applies to any ringed space) gives a canonical isomorphism

$$\mathbf{R}\varGamma_{\overline{\{x\}}}\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,\mathcal{R}^\bullet) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,\mathbf{R}\varGamma_{\overline{\{x\}}}\mathcal{R}^\bullet).$$

Recall that the stalk at x of an injective $\mathcal{O}_{\mathfrak{X}}$-module is an injective $\mathcal{O}_{\mathfrak{X},x}$-module. (In view of [**10**, pp. 110–111, (5.2.6) and (5.2.8)], the proof for schemes in [**11**, p. 128, 7.12] extends to formal schemes.) It follows that $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,-)$ "commutes" with the exact functor "stalk at x." (Since $\mathcal{R}^\bullet$ is a bounded injective complex, one can use [**11**, p. 68, Prop. 7.1] to reduce the proof to the trivial case $\mathcal{F}^\bullet=\mathcal{O}^n_{\mathfrak{X}}$.) Thus there are canonical isomorphisms

$$H^i\mathbf{R}\varGamma_{x}\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,\mathcal{R}^\bullet) \xrightarrow{\ \sim\ } \mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{O}_{\mathfrak{X},x}}(\mathcal{F}^\bullet_x,\mathbf{R}\varGamma_{x}\mathcal{R}^\bullet).$$

Since $\mathbf{R}\varGamma_{x}\mathcal{R}^\bullet \cong \mathcal{R}^\bullet(x)[-\Delta(x)]$ and, by 9.1.3, $\mathcal{R}^\bullet(x)$ is an injective module containing the residue field k_x of $\mathcal{O}_{\mathfrak{X},x}$, therefore

$$\mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{O}_{\mathfrak{X},x}}(\mathcal{F}^\bullet_x,\mathbf{R}\varGamma_{x}\mathcal{R}^\bullet)\cong \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X},x}}\big(\mathrm{H}^{\Delta(x)-i}\mathcal{F}^\bullet_x,\mathcal{R}^\bullet(x)\big).$$

If $\mathrm{H}^{\Delta(x)-i}\mathcal{F}^\bullet_x\neq 0$ then any nonzero principal submodule admits a nonzero $\mathcal{O}_{\mathfrak{X},x}$-homomorphism into $k_x\subset\mathcal{R}^\bullet(x)$, and so, $\mathcal{R}^\bullet(x)$ being injective, there exists a nonzero map $\mathrm{H}^{\Delta(x)-i}\mathcal{F}^\bullet_x\to\mathcal{R}^\bullet(x)$, whence $H^i\mathbf{R}\varGamma_{x}\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,\mathcal{R}^\bullet)\neq 0$. The assertion (9.3.1.1) results. □

Proposition 9.3.5. *Under the assumptions of* 9.3.1 *there are, for any* $n\in\mathbb{Z}$ *and* $\mathcal{F}^\bullet\in\mathbf{D}_{\mathrm{c}}(\mathfrak{X})$, *with* $\mathcal{F}^{\bullet\prime}:=\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,\mathcal{R}^\bullet)$ *and* $H:=H^n\mathcal{F}^\bullet\in\mathbf{D}_{\mathrm{c}}(\mathfrak{X})$, *functorial isomorphisms*

$$E_{\Delta-n}\mathcal{F}^{\bullet\prime} \xrightarrow{\ \sim\ } E_{\Delta-n}\big((H[-n])'\big) \xrightarrow{\ \sim\ } H'[n]\in\mathbf{D}^*_{\mathrm{c}}(\mathfrak{X}).$$

Hence for all $n\in\mathbb{Z}$, $E_{\Delta-n}\mathbf{D}^*_{\mathrm{c}}(\mathfrak{X})\subset\mathbf{D}^*_{\mathrm{c}}(\mathfrak{X})$.

Proof. Let $\theta\colon\mathcal{F}^\bullet_1\to\mathcal{F}^\bullet_2$ be a $\mathbf{D}_{\mathrm{c}}$-map such that $H^n(\theta)$ is an isomorphism $H^n\mathcal{F}^\bullet{}_1\xrightarrow{\ \sim\ }H^n\mathcal{F}^\bullet_2$. As in the proof of 9.3.1, we deduce that for each $x\in\mathfrak{X}$ the map $H^{\Delta(x)-n}_x\mathcal{F}^{\bullet\prime}_1\to H^{\Delta(x)-n}_x\mathcal{F}^{\bullet\prime}_2$ induced by θ is an isomorphism. Thus θ induces an isomorphism $E_{\Delta-n}\mathcal{F}^{\bullet\prime}_1\xrightarrow{\ \sim\ }E_{\Delta-n}\mathcal{F}^{\bullet\prime}_2$.

To get the first isomorphism in the Proposition, apply the preceding to the canonical maps $\theta_1\colon\mathcal{F}^\bullet\to\tau_{\ge n}\mathcal{F}^\bullet$ and $\theta_2\colon(H^n\mathcal{F}^\bullet)[-n]\to\tau_{\ge n}\mathcal{F}^\bullet$, each of which induces a homology isomorphism in degree n.

For the second isomorphism, 9.3.2 gives that H' is a CM-complex in $\mathbf{D}^*_{\mathrm{c}}$, whence

$$E_{\Delta-n}\big((H[-n])'\big)=E_{\Delta-n}\big((H'[n])\big)\underset{(11)}{=}(E_\Delta H')[n]\underset{(3.3.2)}{\xrightarrow{\ \sim\ }}H'[n]\in\mathbf{D}^*_{\mathrm{c}}.$$

The last assertion holds because by the statement preceding Corollary 9.3.2, every $\mathcal{G}\in\mathbf{D}^*_{\mathrm{c}}$ is, isomorphic to an $\mathcal{F}^{\bullet\prime}$. □

Corollary 9.3.6 (cf. [**7**, p.26, Thm. 3.2], [**16**, Thm. 4.4]). *Let* $\mathfrak{X}$, $\mathcal{R}^\bullet$ *and* Δ *be as in* 9.3.1. *Let* $0\neq\mathcal{G}^\bullet\in\mathbf{D}^{*-}_{\mathrm{c}}(\mathfrak{X})$, *so that* $0\neq\mathcal{D}_{\mathrm{t}}\mathcal{G}^\bullet\in\mathbf{D}^+_{\mathrm{c}}(\mathfrak{X})$ *and we can set*

$$m=m(\mathcal{G}^\bullet):=\min\{\,n\mid H^n\mathcal{D}_{\mathrm{t}}\mathcal{G}^\bullet\neq 0\,\}.$$

(i) *There is a canonical* $\mathbf{D}(\mathfrak{X})$*-map* $\mathsf{s}(\mathcal{G}^\bullet)\colon\mathcal{G}^\bullet[-m]\to E_\Delta\big(\mathcal{G}^\bullet[-m]\big)\neq 0$ *such that any* $\mathbf{D}(\mathfrak{X})$*-morphism of* $\mathcal{G}^\bullet[-m]$ *into a* Δ-CM *complex* $\mathcal{E}^\bullet$ *in* $\mathbf{D}^*_{\mathrm{c}}(\mathfrak{X})$ *factors uniquely through* $\mathsf{s}(\mathcal{G}^\bullet)$.

(ii) *If* $\mathcal{G}^\bullet[-m]$ *in* (i) *is* Δ-CM *then* $\mathsf{s}(\mathcal{G}^\bullet)$ *is the isomorphism* $\mathsf{S}(\mathcal{G}^\bullet[-m])$ *of* 3.3.2.

PROOF. For simplicity, we write $\mathcal{G}$ for $\mathcal{G}^\bullet$, ..., $\mathcal{G}'$ for $\mathcal{D}_t\mathcal{G}^\bullet$, ...

Set $\mathcal{F} := \mathcal{G}' \in \mathbf{D}_c(\mathfrak{X})$ and $H := H^m(\mathcal{G})$. By 9.3.5 the natural map $H[-m] \to \mathcal{F}$ induces an isomorphism

$$E_\Delta\big(\mathcal{G}[-m]\big) \underset{(11)}{=} \big(E_{\Delta-m}\mathcal{G}\big)[-m] \xrightarrow{\ \sim\ } H' \ (\neq 0),$$

whose inverse composed with the natural map $\sigma\colon \mathcal{G}[-m] = (\mathcal{F}[m])' \to H'$ is defined to be the map $\mathsf{s}(\mathcal{G})$.

Assertion (i) results now from the sequence of natural isomorphisms

$$\begin{aligned}\operatorname{Hom}\big(\mathcal{G}[-m],\, \mathcal{E}\big) &\cong \operatorname{Hom}\big(\mathcal{E}', \mathcal{G}'[m]\big)\\ &\cong \operatorname{Hom}\big(\mathcal{E}'[-m], \mathcal{F}\big) \cong \operatorname{Hom}\big(\mathcal{E}'[-m], H[-m]\big) \cong \operatorname{Hom}\big(H', \mathcal{E}\big).\end{aligned}$$

(For the third isomorphism, recall from 9.3.1 that $\mathcal{E}'$ is a coherent $\mathcal{O}_\mathfrak{X}$-module.)

If $\mathcal{G}[-m]$ is Δ-CM then by 9.3.1, $\mathcal{F}[m]$ is a coherent $\mathcal{O}_\mathfrak{X}$-module, so that σ is an isomorphism, whence so is $\mathsf{s}(\mathcal{G})$; and then functoriality of S forces $\mathsf{s}(\mathcal{G}) = \mathsf{S}\big(\mathcal{G}[-m]\big)$, proving (ii). $\square$

EXAMPLE 9.3.7. Suppose that in 9.3.6, $\mathfrak{X}$ is an *ordinary* scheme. After translating $\mathcal{R}$ we may assume that $m(\mathcal{O}_\mathfrak{X}) = 0$. Then with $K := H^0(\mathcal{O}'_\mathfrak{X}) = H^0(\mathcal{R})$, a *canonical* $\mathcal{O}_\mathfrak{X}$*-module,* 9.3.5 for $\mathcal{F}^\bullet = \mathcal{R}$ gives $E_\Delta(\mathcal{O}_\mathfrak{X}) \cong K'$, whence $m(K) = 0$. If $\mathcal{O}_\mathfrak{X}$ satisfies the Serre condition (S_2) then the natural map is an isomorphism

$$\mathcal{O}_\mathfrak{X} \xrightarrow{\ \sim\ } H^0(E_\Delta(O_\mathfrak{X})) \cong H^0(K') \ (= \mathcal{H}om(K,K));$$

so in this case 9.3.5 for $\mathcal{F}^\bullet = K'$ gives

$$E_\Delta(K) \cong E_\Delta(K'') \cong \mathcal{O}'_\mathfrak{X} = \mathcal{R}.$$

Similar considerations apply with an arbitrary coherent $\mathcal{O}_\mathfrak{X}$-module in place of $\mathcal{O}_\mathfrak{X}$, see, e.g., [**6**, Thm. 4.6 (and 1.4)].

Keeping the notation and assumptions of 9.3.6 and its proof, we close with some characterizations of the integer $m(\mathcal{G})$.

Replacing m by $n < m$ in the proof of 9.3.6(i) gives that $E_\Delta\big(\mathcal{G}[-n]\big) = E_{\Delta-n}\mathcal{G} = 0$, and further that any $\mathbf{D}(\mathfrak{X})$-morphism of $\mathcal{G}[-n]$ into a Δ-CM complex in $\mathbf{D}_c^*(\mathfrak{X})$ vanishes. The proof of 9.3.1 shows further that with $\mathcal{F} := \mathcal{G}'$, $H_x^{\Delta(x)-m}\mathcal{G} \neq 0 \iff x \in \operatorname{Supp}(H^m\mathcal{F})$. In particular, $m := m(\mathcal{G})$ *is the unique integer such that*

(a) *for all* $x \in \mathfrak{X}$ *and* $q > \Delta(x) - m$, $H_x^q\mathcal{G} = 0$, *and*

(b) *there is an* $x \in \mathfrak{X}$ *such that* $H_x^{\Delta(x)-m}\mathcal{G} \neq 0$.

We claim that, furthermore:

(1) $m = m' := \min\{\, \Delta(x) - q \mid q \in \mathbb{Z},\ x \in \operatorname{Supp}(H^q\mathcal{G})\,\}$, *and*

(2) *for any generic point* x *of* $\operatorname{Supp}(H^m\mathcal{F})$, $x \in \operatorname{Supp}(H^q\mathcal{G})$ *and* $\Delta(x) - q = m$.

In less contorted terms, $m := m(\mathcal{G})$ *is the unique integer such that*

(a)$'$ *for all* $x \in \mathfrak{X}$ *and* $q > \Delta(x) - m$, $\mathrm{H}^q(\mathcal{G}_x) = 0$, *and*

(b)$'$ *for any generic point* x *of* $\operatorname{Supp}(H^m\mathcal{F})$, $\mathrm{H}^{\Delta(x)-m}(\mathcal{G}_x) \neq 0$.

PROOF. Let $Z^\bullet$ be the filtration defined by Δ (i.e., $x \in Z^p \iff \Delta(x) \geq p$.) In view of (a), ($8'$) gives, for any $j > p - m$,

$$H^j_{Z^p/Z^{p+1}}\mathcal{G} = \bigoplus_{\Delta(x)=p} i_x(H_x^j\mathcal{G}) = 0.$$

So if $q + m > d := \min_{x\in\mathfrak{X}} \Delta(x)$, there is a surjection followed by isomorphisms

$$(*) \qquad H^q_{Z^{q+m}}\mathcal{G} \twoheadrightarrow H^q_{Z^{q+m-1}}\mathcal{G} \xrightarrow{\ \sim\ } H^q_{Z^{q+m-2}}\mathcal{G} \xrightarrow{\ \sim\ } \cdots \xrightarrow{\ \sim\ } H^q_{Z^d}\mathcal{G} = H^q\mathcal{G}.$$

Hence $\operatorname{Supp}(H^q\mathcal{G}) \subset \operatorname{Supp}(H^q_{Z^{q+m}}\mathcal{G}) \subset Z^{q+m}$, and so $m \leq m'$.

Next, among points in $\mathrm{Supp}(H^m\mathcal{F})$ choose one, say x, where the value of Δ is minimal. We will show that $H^{\Delta(x)-m}\mathcal{G}_x \neq 0$, from which follows that $m' \le m$, proving (1).

It also follows that if z is *any* generic point of $\mathrm{Supp}(H^m\mathcal{F})$ then, since z has an open neighborhood V within which z is the *only* such generic point, therefore

$$0 \neq H^{\Delta(z)-m(\mathcal{G}|_V)}(\mathcal{G}|_V)_z = H^{\Delta(z)-m}\mathcal{G}_z,$$

proving (2).

Set $\delta := \Delta(x)$. As before, $H_x^{\delta-m}\mathcal{G}$ is dual to $\mathrm{H}^m\mathcal{F}_x \neq 0$, so $H_x^{\delta-m}\mathcal{G} \neq 0$. Similarly, if $\Delta(y) < \delta$, so that $y \notin \mathrm{Supp}(H^m\mathcal{F})$, then $H_y^{\Delta(y)-m}\mathcal{G} = 0$. Hence, with $q := \delta - m$,

(i) $H^q_{Z^{q+m}/Z^{q+m+1}}\mathcal{G}$ has nonzero stalk at x; and

(ii) $H^{q-1}_{Z^{q+m-1}/Z^{q+m}}\mathcal{G} = 0$.

From (ii) it follows that the natural map $H^q_{Z^{q+m}}\mathcal{G} \to H^q_{Z^{q+m-1}}\mathcal{G}$ is injective. Then since $q \ge d - m$, $(*)$ shows that $H^q_{Z^{q+m}}\mathcal{G} \cong H^q\mathcal{G}$ (the case $q = d - m$ being trivial).

Since for any abelian sheaf A, the stalk at x of $\varGamma_{Z^{\delta+1}}A$ vanishes, therefore the natural map is an isomorphism $(H^q_{Z^{q+m}}\mathcal{G})_x \xrightarrow{\sim} (H^q_{Z^\delta/Z^{\delta+1}}\mathcal{G})_x$; so for nonvanishing of $H^{\Delta(x)-m}\mathcal{G}_x$ we need only note that by (i), the target of this isomorphism doesn't vanish. $\square$

10. Some explicit descriptions

We conclude this paper by giving explicit descriptions of some notions encountered in earlier sections. In §10.1 we concentrate on the Suominen isomorphism for Cohen-Macaulay complexes; this is the isomorphism $1_{\overline{\mathbf{D}}} \xrightarrow{\sim} QE$ defined in 3.3.2. There it is defined in an implicit way. Here we give an explicit description for it in several cases. (See also Corollary 9.3.6(ii).) In §10.3 we study $f^\sharp$ for an étale map f. This includes the case of completion of a formal scheme along an open ideal and also the case of open immersions which we covered in 8.1.1 and 8.2.1.

We use the following notation throughout this section.

(a) Let $(\mathcal{X}, \Delta) \in \mathbb{F}_{\mathrm{c}}$. Let $\overline{\mathbf{D}} := \mathbf{D}^+(\mathcal{X}, \Delta)_{\mathrm{CM}}$ be the category of Δ-CM complexes on $\mathcal{X}$ (§3.3). Then the *Suominen isomorphism* $\mathsf{S}_{\mathcal{X},\Delta}$ (or simply S in case of no ambiguity) is the isomorphism $1_{\overline{\mathbf{D}}} \xrightarrow{\sim} QE$ defined in 3.3.2. For any $\mathcal{G}^\bullet \in \overline{\mathbf{D}}$ we denote the induced isomorphism $\mathcal{G}^\bullet \xrightarrow{\sim} QE\mathcal{G}^\bullet$ by $\mathsf{S}(\mathcal{G}^\bullet)$.

(b) For any smooth map $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ in $\mathbb{F}_{\mathrm{c}}$ of relative dimension d, we use $f^\diamond(-) := f^*(-) \otimes_{\mathcal{X}} \omega_f[d]$.

10.1. The Suominen isomorphism. Let $(\mathcal{X}, \Delta) \in \mathbb{F}_{\mathrm{c}}$. Let $\mathcal{G}^\bullet$ be a Δ-CM complex on $\mathcal{X}$ with bounded homology. In [**11**, p. 242-246], Hartshorne describes a non-canonical way of constructing an isomorphism $\mathcal{G}^\bullet \xrightarrow{\sim} QE\mathcal{G}^\bullet$. This construction lacks functorial properties. Nevertheless it satisfies a property, recalled below, that makes the isomorphism functorial when $\mathcal{G}^\bullet$ ranges over *Gorenstein* complexes ([**11**, p. 248]). We now show that the Suominen isomorphism $\mathsf{S}(\mathcal{G}^\bullet)$ also satisfies this property, so that if $\mathcal{G}^\bullet$ is Gorenstein, then $\mathsf{S}(\mathcal{G}^\bullet)$ agrees with $\phi(\mathcal{G}^\bullet)$ for $\phi\colon 1 \xrightarrow{\sim} QE$ as constructed in [**11**, p. 249].

Let $(\mathcal{X}, \Delta) \in \mathbb{F}_{\mathrm{c}}$. Let $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathcal{X})$, and let $\mathcal{L}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. Let $Z^\bullet$ denote the filtration of $\mathcal{X}$ induced by Δ. Let $E_r^{p,q}(\mathcal{F}^\bullet) = E_r^{p,q}(\mathcal{L}^\bullet)$ be the spectral sequence associated to the filtered complex $\varGamma_{Z^\bullet}\mathcal{L}^\bullet$. Recall that $E_\Delta\mathcal{F}^\bullet$ is given by the $E_1^{*,0}$-terms of this spectral sequence. Now if $\mathcal{F}^\bullet$ is also Δ-CM, then the $E_1^{p,q}$-terms vanish for $q \neq 0$ so that the spectral sequence degenerates. Since the spectral sequence converges to the homology of $\mathcal{L}^\bullet$, there results, for any $i \in \mathbb{Z}$, (with $\mathcal{F}^\bullet$ Δ-CM) a natural isomorphism

$$\psi^i_{\mathcal{F}^\bullet}\colon H^iE_\Delta\mathcal{F}^\bullet = E_2^{i,0}(\mathcal{L}^\bullet) \xrightarrow{\sim} H^i\mathcal{L}^\bullet \xrightarrow{\sim} H^i\mathcal{F}^\bullet.$$

Let $\overline{\mathbf{C}}, \overline{\mathbf{D}}, Q, E$ be as in §3.3. Set $\overline{\mathbf{D}}^{\mathrm{b}} := \overline{\mathbf{D}} \cap \mathbf{D}^{\mathrm{b}}(\mathcal{X})$.

Proposition 10.1.1. *With notation as above, for any $\mathcal{F}^\bullet \in \overline{\mathbf{D}}^{\mathrm{b}}$ and for any integer i, the induced isomorphism $H^i\mathsf{S}(\mathcal{F}^\bullet)$ is inverse to $\psi^i_{\mathcal{F}^\bullet}$.*

Proof. Since $H^i\mathsf{S}(-)$ and $\psi^i_{(-)}$ are functorial in $\mathcal{F}^\bullet$ we may replace $\mathcal{F}^\bullet$ by any isomorphic complex in $\overline{\mathbf{D}}^{\mathrm{b}}$. In particular, since $\mathcal{F}^\bullet$ is isomorphic to a Cousin complex, therefore, we may assume without loss of generality that $\mathcal{F}^\bullet = Q\mathcal{C}^\bullet$ where $\mathcal{C}^\bullet$ is a Cousin complex.

By 3.3.2, the isomorphism $\mathsf{S}(Q\mathcal{C}^\bullet)\colon Q\mathcal{C}^\bullet \xrightarrow{\sim} QEQ\mathcal{C}^\bullet$ is the same as the Q-image of the inverse of the isomorphism $\phi_{\mathcal{C}^\bullet}\colon EQ\mathcal{C}^\bullet \xrightarrow{3.2.1(\mathrm{ii})} \mathcal{C}^\bullet$. Therefore it suffices to show that $\psi^i_{\mathcal{C}^\bullet} = H^i\phi_{\mathcal{C}^\bullet}$.

Before proceeding further, we need a definition. For any complex $\mathcal{G}^\bullet$ on $\mathcal{X}$, let $\mathsf{E}\mathcal{G}^\bullet$ denote the complex given by the $E_1^{*,0}$-terms of the spectral sequence associated to the filtered complex $\varGamma_{Z^\bullet}\mathcal{G}^\bullet$; the only difference from the definition of $E\mathcal{G}^\bullet$ being that now we do not replace $\mathcal{G}^\bullet$ by an injective resolution. If $\mathcal{G}^\bullet$ consists of flasque

sheaves, then there is a canonical isomorphism $\mathsf{E}\mathcal{G}^\bullet \xrightarrow{\sim} E\mathcal{G}^\bullet$. If $\mathcal{G}^\bullet$ is Δ-Cousin, then $\mathsf{E}\mathcal{G}^\bullet = \mathcal{G}^\bullet$ (3.2.1(ii)) and moreover the filtration $\{\varGamma_{Z^n}\mathcal{G}^\bullet\}_{n\in\mathbb{Z}}$ is now given by truncations (3.2.1(i)[a]). Therefore the $E_1^{p,q}$-terms of the associated spectral sequence vanish for $q \neq 0$. It follows that if $\mathcal{G}^\bullet$ is a Δ-Cousin complex, then the canonical maps

$$H^i\mathcal{G}^\bullet = H^i\mathsf{E}\mathcal{G}^\bullet \xrightarrow{\text{deg. sp. seq. for } \varGamma_{Z^\bullet}\mathcal{G}^\bullet} H^i\mathcal{G}^\bullet$$

compose to the identity map.

Returning to the proof, let $\mathcal{C}^\bullet \to \mathcal{L}^\bullet$ be an injective resolution. Consider the following commutative diagram of natural isomorphisms.

$$\begin{tikzcd}
H^i\mathcal{C}^\bullet \arrow[r, "\alpha", "\mathcal{C}^\bullet = \mathsf{E}\,\mathcal{C}^\bullet"'] & H^i\mathsf{E}\,\mathcal{C}^\bullet \arrow[r, "\text{deg. sp. seq. for } \varGamma_{Z^\bullet}\mathcal{C}^\bullet"'] \arrow[d, "\beta"'] & H^i\mathcal{C}^\bullet \arrow[d, "\gamma"] \\
& H^i\mathsf{E}\,\mathcal{L}^\bullet \arrow[r, "\delta", "\text{deg. sp. seq. for } \varGamma_{Z^\bullet}\mathcal{L}^\bullet"'] & H^i\mathcal{L}^\bullet
\end{tikzcd}$$

By definition, $\mathsf{E}\,\mathcal{L}^\bullet = E\mathcal{C}^\bullet$ and moreover $(\beta\alpha)^{-1} = H^i\phi_{\mathcal{C}^\bullet}$ and $\gamma^{-1}\delta = \psi^i_{\mathcal{C}^\bullet}$. Since the top row composes to the identity map, it follows that $H^i\phi_{\mathcal{C}^\bullet} = \psi^i_{\mathcal{C}^\bullet}$. $\square$

Our next goal concerns adic smooth maps in $\mathbb{F}_c$. We begin with a preliminary result on fibers.

For any map of formal schemes $f\colon \mathcal{X} \to \mathcal{Y}$, the fiber of f over a point $y \in \mathcal{Y}$ is the formal scheme $\mathcal{X}_y := \mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(k(y))$ where $k(y)$ is the residue field of $\mathcal{O}_{\mathcal{Y},y}$. As is the usual practice, we shall identify the underlying topological space of $\mathcal{X}_y$ with its canonical image in $\mathcal{X}$.

As usual, for points $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we shall denote the maximal ideals at $\mathcal{O}_{\mathcal{X},x}, \mathcal{O}_{\mathcal{Y},y}$ by m_x, m_y respectively. The following basic facts on fiber spaces follow easily from the definitions. For any $x \in \mathcal{X}_y$, the stalk of the structure sheaf of $\mathcal{X}_y$ at x is canonically isomorphic to $\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x}$. If f is adic, then $\mathcal{X}_y$ is an ordinary scheme.

Lemma 10.1.2. *Let $f\colon (\mathcal{X},\Delta_1) \to (\mathcal{Y},\Delta_2)$ be a smooth map in $\mathbb{F}_c$ of constant relative dimension d. Let $y \in \mathcal{Y}$. Then for any $x \in \mathcal{X}_y$, $0 \le \Delta_2(y) - \Delta_1(x) \le d$. Moreover, if f is adic, then the following conditions are equivalent.*

(i) *x is a generic point of $\mathcal{X}_y$.*
(ii) *$m_y\mathcal{O}_{\mathcal{X},x} = m_x$.*
(iii) *$\Delta_1(x) = \Delta_2(y) - d$.*

Proof. By definition, $\Delta_1 = f^\sharp\Delta_2$ (2.1.2) so that $\Delta_2(y) - \Delta_1(x)$ equals the transcendence degree of the residue field extension $k(y) \to k(x)$ and hence is non-negative. The other inequality follows from 2.6.10 which also immediately implies (ii) $\iff$ (iii). If (iii) holds, then for any nontrivial specialization $x' \rightsquigarrow x$ we get a contradiction because $\Delta_1(x') < \Delta_1(x) = \Delta_2(y) - d$. Thus (iii) $\Longrightarrow$ (i).

If f is adic and smooth, then $\mathcal{X}_y$ is a smooth ordinary scheme over $\mathrm{Spec}(k(y))$ and hence is a disjoint union of integral schemes. Therefore if x is a generic point of $\mathcal{X}_y$ then the local ring R of $\mathcal{X}_y$ at x is a field. In view of the canonical isomorphism $R \xrightarrow{\sim} \mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x}$, (i) $\Longrightarrow$ (ii) follows. $\square$

Let $f\colon (\mathcal{X},\Delta_{\mathcal{X}}) \to (\mathcal{Y},\Delta_{\mathcal{Y}})$ be an *adic* smooth map in $\mathbb{F}_c$ of constant relative dimension d. Let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Since $\mathcal{M}^\bullet$ consists of torsion

$\mathcal{O}_{\mathcal{Y}}$-modules, f being adic implies that $f^*\mathcal{M}^\bullet$ consists of torsion $\mathcal{O}_{\mathcal{X}}$-modules. In particular, $f^\diamond\mathcal{M}^\bullet = f^*(-)\otimes_{\mathcal{X}}\omega_f[d]$ also consists of torsion $\mathcal{O}_{\mathcal{X}}$-modules and hence the canonical $\mathbf{D}(\mathcal{X})$-map $\mathbf{R}\varGamma'_{\mathcal{X}} f^\diamond\mathcal{M}^\bullet \to f^\diamond\mathcal{M}^\bullet$ is an isomorphism. Therefore, by 5.1.3, $f^\diamond\mathcal{M}^\bullet$ is a Δ_x-CM complex. Set $E := E_{\Delta_{\mathcal{X}}}$. Our aim is twofold :

- To explicitly construct a quasi-isomorphism $\eta_f(\mathcal{M}^\bullet)\colon f^\diamond\mathcal{M}^\bullet \to Ef^\diamond\mathcal{M}^\bullet$ in $\mathbf{C}(\mathcal{X})$.
- To show that the natural image of $\eta_f(\mathcal{M}^\bullet)$ in $\mathbf{D}(\mathcal{X})$ is the same as the Suominen isomorphism $\mathsf{S}(f^\diamond\mathcal{M}^\bullet)$.

Before proceeding further we need to set up some notation. Much of the notation concerns the fact that both $f^\diamond\mathcal{M}^\bullet$ and $Ef^\diamond\mathcal{M}^\bullet$ also have a graded decomposition arising from the punctual grading on $\mathcal{M}^\bullet$, i.e., one parametrized by the points of $\mathcal{Y}$. For the rest of this subsection, we shall assume that $f, \mathcal{X}, \Delta_x, \mathcal{Y}, \Delta_y, d$ are fixed. Set $\omega := \omega_f$. Here then is some notation that we shall use for the rest of this subsection.

(i) For any $y \in \mathcal{Y}$ with $M = \mathcal{M}^\bullet(y)$, set $f^\diamond(\mathcal{M}^\bullet, y) := f^* i_y M \otimes_{\mathcal{Y}} \omega$. For any integer a, there results natural isomorphism

$$(f^\diamond\mathcal{M}^\bullet)^a \xrightarrow{\ \sim\ } \bigoplus_{\{y\in\mathcal{Y}\,|\,\Delta_{\mathcal{Y}}(y)=a+d\}} f^\diamond(\mathcal{M}^\bullet, y).$$

(ii) For any $x \in \mathcal{X}$, with $y = f(x), p = \Delta_x(x) - \Delta_y(y) + d, M = \mathcal{M}^\bullet(y)$, set

$$G_{x,\mathcal{M}^\bullet} := H^p_{m_x}(M \otimes_y \omega_x).$$

(iii) For any $y \in \mathcal{Y}$ and any integer a, set $\mathcal{E}^a_{\mathcal{M}^\bullet}(y) := \bigoplus_x i_x(G_{x,\mathcal{M}^\bullet})$ where x varies over points in $\mathcal{X}_y$ such that $\Delta_{\mathcal{X}}(x) = a$.

(iv) For any integers a, b set

$$\mathcal{E}^{a,b}_{\mathcal{M}^\bullet} := \bigoplus_{\{\,y\in\mathcal{Y}\,|\,\Delta_{\mathcal{Y}}(y)=b\}} \mathcal{E}^a_{\mathcal{M}^\bullet}(y) \quad = \bigoplus_{\{\,x\in\mathcal{X}\,|\,\Delta_{\mathcal{X}}(x)=a,\ \Delta_{\mathcal{Y}}(f(x))=b\}} i_x(G_{x,\mathcal{M}^\bullet}).$$

By 10.1.2, if $b < a$ or $b > a + d$, then $\mathcal{E}^{a,b}_{\mathcal{M}^\bullet}$ ($= \mathcal{E}^{a,b}$ for simplicity) has an empty sum and hence equals 0. The points contributing to $\mathcal{E}^{a,a+d}$ are ones that are generic in their fiber while those contributing to $\mathcal{E}^{a,a}$ are ones that are closed in their fiber.

(v) For any integer a set

$$\mathcal{E}^a_{\mathcal{M}^\bullet} := \bigoplus_{b\in\mathbb{Z}} \mathcal{E}^{a,b}_{\mathcal{M}^\bullet} \quad = \quad \mathcal{E}^{a,a} \oplus \mathcal{E}^{a,a+1} \oplus \cdots \oplus \mathcal{E}^{a,a+d}.$$

Let us first record that for any integer a there is a natural isomorphism

$$(Ef^\diamond\mathcal{M}^\bullet)^a \xrightarrow{\ \sim\ } \mathcal{E}^a_{\mathcal{M}^\bullet}, \tag{87}$$

defined as follows. Let $x \in \mathcal{X}$ be such that $\Delta_{\mathcal{X}}(x) = a$. For $y = f(x), M = \mathcal{M}^\bullet(y)$, $q = \Delta_{\mathcal{Y}}(y)$ and $p_1 = a - q + d$, consider the natural isomorphism

$$(Ef^\diamond\mathcal{M}^\bullet)(x) \xrightarrow{(-1)^{qd} \text{ times } (38)} H^{p_1}_{m_x}(M \otimes_y \omega_x) = G_{x,\mathcal{M}^\bullet}.$$

By summing up over all $x \in \mathcal{X}$ such that $\Delta_{\mathcal{X}}(x) = a$, we obtain (87).

Our next step is to define, for any integer a, a canonical map $(f^\diamond\mathcal{M}^\bullet)^a \to \mathcal{E}^a_{\mathcal{M}^\bullet}$. Let $y \in \mathcal{Y}$ be such that $\Delta_{\mathcal{Y}}(y) = a + d$. Let $x \in \mathcal{X}_y$ be such that $\Delta_{\mathcal{X}}(x) = a$.

By 10.1.2, x is generic in the fiber and $m_y\mathcal{O}_{\mathcal{X},x} = m_x$. Hence, with $M = \mathcal{M}^\bullet(y)$ we obtain

$$(88)\qquad (f^\diamond(\mathcal{M}^\bullet, y))_x \cong M \otimes_y \omega_x = \varGamma_{m_x}(M \otimes_y \omega_x) = H^0_{m_x}(M \otimes_y \omega_x) = G_{x,\mathcal{M}^\bullet}$$

where the first of the three equalities holds because m_y-torsionness of M implies that $M \otimes_y \omega_x$ is $m_y\mathcal{O}_{\mathcal{X},x}$-torsion. For a fixed y, by varying x over all the generic points of $\mathcal{X}_y$ (equivalently, over points in $\mathcal{X}_y$ such that $\Delta_{\mathcal{X}}(x) = a$) we therefore obtain a map

$$f^\diamond(\mathcal{M}^\bullet, y) \xrightarrow{\text{canonical}} \bigoplus_x i_x(f^\diamond(\mathcal{M}^\bullet, y))_x \xrightarrow{(88)} \bigoplus_x i_x G_{x,\mathcal{M}^\bullet} = \mathcal{E}^a_{\mathcal{M}^\bullet}(y).$$

Finally, taking direct sums over all $y \in \mathcal{Y}$ such that $\Delta_{\mathcal{Y}}(y) = a + d$, we obtain a map

$$(89)\qquad (f^\diamond\mathcal{M}^\bullet)^a \longrightarrow \mathcal{E}^{a,a+d}_{\mathcal{M}^\bullet} \hookrightarrow \mathcal{E}^a_{\mathcal{M}^\bullet}.$$

Combining (87) with (89) results in a natural graded map

$$(90)\qquad \eta_f(\mathcal{M}^\bullet)\colon f^\diamond\mathcal{M}^\bullet \longrightarrow Ef^\diamond\mathcal{M}^\bullet.$$

The rest of this subsection is devoted to showing that (90) is a map of complexes, in fact a quasi-isomorphism whose derived-category image is the Suominen isomorphism $\mathsf{S}(f^\diamond\mathcal{M}^\bullet)$. We first show this in the special case when $\mathcal{M}^\bullet$ is concentrated in one spot (10.1.5, 10.1.6) and in 10.1.8, 10.1.9, the general case is considered.

Lemma 10.1.3. *Let $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ be an* adic *smooth map in $\mathbb{F}_c$ of constant relative dimension d. Let $Z^\bullet$ be the filtration on $\mathcal{X}$ induced by $\Delta_{\mathcal{X}}$. Let $\mathcal{M}^\bullet$ be a complex in $\mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$ such that $\mathcal{M}^\bullet$ is concentrated in only one degree, so that $\mathcal{N}^\bullet := f^\diamond\mathcal{M}^\bullet$ is also concentrated in one degree, say a (in other words, $\mathcal{N}^\bullet = \mathcal{N}^a[-a]$ and $\mathcal{M}^\bullet = \mathcal{M}^{a+d}[-a-d]$). For any $y \in \mathcal{Y}$, set $\mathcal{N}(y) := f^\diamond(\mathcal{M}^\bullet, y)$. Then the following hold.*

(i) $H^a_{Z^a}\mathcal{N}^\bullet = \varGamma_{Z^a}\mathcal{N}^a = \mathcal{N}^a = H^a\mathcal{N}^\bullet$.

(ii) $\mathcal{E}^n_{\mathcal{M}^\bullet} = 0$ *for $n < a$ or $n > a + d$.*

(iii) $\mathcal{E}^{a,a+d}_{\mathcal{M}^\bullet} = \mathcal{E}^a_{\mathcal{M}^\bullet}$ *or, in other words, $\mathcal{E}^{a,b}_{\mathcal{M}^\bullet} = 0$ for $b \neq a + d$. Moreover, for any $x \in \mathcal{X}$ such that $\Delta_{\mathcal{X}}(x) = a$ and $\Delta_{\mathcal{Y}}(f(x)) = b \neq a + d$, we also have*

$$H^a_x\mathcal{N}^\bullet = H^0_x\mathcal{N}^a = \varGamma_x\mathcal{N}^a = 0.$$

(iv) *Let $x \in \mathcal{X}$, $y = f(x)$ and $M = \mathcal{M}^\bullet(y)$. If $\Delta_{\mathcal{X}}(x) = a$ and $\Delta_{\mathcal{Y}}(y) = a + d$ then*

$$M \otimes_y \omega_x \cong (\mathcal{N}(y))_x = \varGamma_x(\mathcal{N}(y)) = \varGamma_x\mathcal{N}^a = \mathcal{N}^a_x.$$

Proof. (i). Since $\mathcal{N}^\bullet = \mathcal{N}^a[-a]$, hence $H^a_{Z^a}\mathcal{N}^\bullet = \varGamma_{Z^a}\mathcal{N}^a$ and $\mathcal{N}^a = H^a\mathcal{N}^\bullet$. Hence the natural map $\gamma\colon H^a_{Z^a}\mathcal{N}^\bullet \to H^a\mathcal{N}^\bullet$ is injective. Since $\mathcal{N}^\bullet$ is $\Delta_{\mathcal{X}}$-CM (5.1.3), therefore by definition of Cohen-Macaulayness, γ is also surjective.

(ii). If $n < a$ or $n > a + d$, then for any $x \in \mathcal{X}$ such that $\Delta_{\mathcal{X}}(x) = n$, we have $\Delta_{\mathcal{Y}}(f(x)) \neq a + d$ by 10.1.2. Since $\mathcal{M}^\bullet$ is concentrated in degree $a + d$ only, hence $\mathcal{M}^\bullet(f(x)) = 0$. Thus $G_{x,\mathcal{M}^\bullet} = 0$ and the result follows.

(iii). Let $b \neq a + d$. Fix $x \in \mathcal{X}$ such that $\Delta_{\mathcal{X}}(x) = a$ and $\Delta_{\mathcal{Y}}(f(x)) = b$. In the canonical decomposition $\mathcal{N}^a = \oplus_y\mathcal{N}(y)$, we have $y \neq f(x)$ for all y contributing in the summation, since y only ranges over points with codimension $a + d$. Hence, by 3.1.11, $\mathbf{R}\varGamma_x\mathcal{N}^a = 0$. The desired conclusion follows.

(iv). Since $\mathcal{N}(y)$ is an $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-module, the first two isomorphisms follow from (88) and 3.1.3. For any other point $y' \in \mathcal{Y}$ such that $\Delta_{\mathcal{Y}}(y') = a+d$ there is no specialization $y' \rightsquigarrow y$ and hence $(i_{y'}M')_y = 0$ for any $\mathcal{O}_{\mathcal{Y},y'}$-module M'. In particular, $(\mathcal{N}(y'))_x = (f^* i_{y'}(\mathcal{M}^\bullet(y')) \otimes_{\mathcal{X}} \omega)_x = 0$. In view of the canonical decomposition of $\mathcal{N}^a$, the last two equalities of (iv) result. $\square$

Lemma 10.1.4. *Let notation and assumptions be as in* 10.1.3. *Then the map*

$$\psi\colon (f^\diamond \mathcal{M}^\bullet)^a = \mathcal{N}^a = H^a_{Z^a}(\mathcal{N}^\bullet) \xrightarrow{\textit{canonical}} H^a_{Z^a/Z^{a+1}}(\mathcal{N}^\bullet) = (E\mathcal{N}^\bullet)^a = (Ef^\diamond\mathcal{M}^\bullet)^a$$

equals the map of (90) *in degree* a.

Proof. Consider the following diagram wherein $\oplus_y, \oplus_x$ and $\oplus_z$ are indexed over the sets $\{y \in \mathcal{Y} | \Delta_{\mathcal{Y}}(y) = a+d\}$, $\{x \in \mathcal{X}_y | \Delta_{\mathcal{X}}(x) = a\}$, and $\{z \in \mathcal{X} | \Delta_{\mathcal{X}}(z) = a\}$ respectively.

$$\begin{tikzcd}
 & (Ef^\diamond\mathcal{M}^\bullet)^a \arrow[d, equal] & & \\
H^a_{Z^a}\mathcal{N}^\bullet \arrow[r] \arrow[d, equal] & H^a_{Z^a/Z^{a+1}}\mathcal{N}^\bullet \arrow[d, equal] & & \\
H^0_{Z^a}\mathcal{N}^a \arrow[r] \arrow[d, equal] & H^0_{Z^a/Z^{a+1}}\mathcal{N}^a \arrow[r, "\sim"] \arrow[d, phantom, "\square_1"] & \oplus_z i_z H^0_z\mathcal{N}^a \arrow[d, equal] \\
(f^\diamond\mathcal{M}^\bullet)^a = \mathcal{N}^a \arrow[r] \arrow[d, "\wr"] & \oplus_y \oplus_x i_x \mathcal{N}^a_x \arrow[r, equal] \arrow[dr, phantom, "\square_2"] & \oplus_z i_z \varGamma_z \mathcal{N}^a \\
\oplus_y \mathcal{N}(y) \arrow[r] & \oplus_y \oplus_x i_x(\mathcal{N}(y))_x \arrow[r, equal] \arrow[u, "\wr"] \arrow[d, "\wr"] \arrow[dr, phantom, "\square_3"] & \oplus_y \oplus_x i_x \varGamma_x(\mathcal{N}(y)) \arrow[u, "\wr"] \arrow[d, "\wr"] \\
 & \oplus_y \oplus_x i_x(M_y \otimes_y \omega_x) \arrow[r, equal] & \oplus_y \oplus_x i_x \varGamma_{m_x}(M_y \otimes_y \omega_x)
\end{tikzcd}$$

The identifications in $\square_2$ and $\square_3$ follow from 10.1.3(iii), (iv). Rest of the maps are the obvious canonical ones. The commutativity of $\square_1$ follows from the description of the punctual decomposition in (8) while that of the remaining rectangles is straightforward. To prove the Lemma, it suffices to check that both, ψ and the degree a component of (90), occur as the composition over the two different paths along the outer border from $(f^\diamond\mathcal{M}^\bullet)^a$ to $(Ef^\diamond\mathcal{M}^\bullet)^a$ in the above diagram. The case of ψ is clear, and the other one follows upon examining the definitions involved. $\square$

Proposition 10.1.5. *Let notation and assumptions be as in* 10.1.3. *Then the graded map* $\eta_f(\mathcal{M}^\bullet)\colon \mathcal{N}^\bullet \to E\mathcal{N}^\bullet$ *of* (90) *is a quasi-isomorphism of complexes.*

Proof. By assumption, $\mathcal{N}^\bullet$ is concentrated only in degree a and in view of the isomorphism (87), by 10.1.3(ii) it follows that $(E\mathcal{N}^\bullet)^n = 0$ for $n < a$.

Since $\mathcal{N}^\bullet$ is $\Delta_{\mathcal{X}}$-CM, therefore $\mathcal{N}^\bullet$ and $E\mathcal{N}^\bullet$ are isomorphic in $\mathbf{D}(\mathcal{X})$ say, via $\mathsf{S}(\mathcal{N}^\bullet)$, and hence to prove the Proposition it suffices to show that in (90), $\mathcal{N}^a$ is

mapped bijectively to the kernel of $\delta = \delta^a_{E\mathcal{N}^\bullet} \colon H^a_{Z^a/Z^{a+1}}\mathcal{N}^\bullet \to H^{a+1}_{Z^{a+1}/Z^{a+2}}\mathcal{N}^\bullet$. By 10.1.4, it suffices to show that the sequence of canonical maps

$$H^a_{Z^a}\mathcal{N}^\bullet \to H^a_{Z^a/Z^{a+1}}\mathcal{N}^\bullet \xrightarrow{\delta} H^{a+1}_{Z^{a+1}/Z^{a+2}}\mathcal{N}^\bullet$$

is exact.

By Cohen-Macaulayness of $\mathcal{N}^\bullet$ we have $H^a_{Z^{a+1}/Z^{a+2}}\mathcal{N}^\bullet = 0$. Since δ corresponds to the ath connecting homomorphism in the cohomology long exact sequence associated to the exact sequence

$$0 \longrightarrow \varGamma_{Z^{a+1}/Z^{a+2}} \longrightarrow \varGamma_{Z^a/Z^{a+2}} \longrightarrow \varGamma_{Z^a/Z^{a+1}} \longrightarrow 0,$$

we see that $\ker(\delta) = H^a_{Z^a/Z^{a+2}}\mathcal{N}^\bullet$. Therefore it suffices to show that the natural map $H^a_{Z^a}\mathcal{N}^\bullet \to H^a_{Z^a/Z^{a+2}}\mathcal{N}^\bullet$ is an isomorphism. Since $\mathcal{N}^\bullet$ is CM, therefore $H^a_{Z^{a+2}}\mathcal{N}^\bullet = H^{a+1}_{Z^{a+2}}\mathcal{N}^\bullet = 0$. Therefore from the cohomology long exact sequence associated to the exact sequence

$$0 \longrightarrow \varGamma_{Z^{a+2}} \longrightarrow \varGamma_{Z^a} \longrightarrow \varGamma_{Z^a/Z^{a+2}} \longrightarrow 0$$

we obtain the desired conclusion. □

Proposition 10.1.6. *Let notation and assumptions be as in* 10.1.5. *Then the derived-category image of* $\eta_f(\mathcal{M}^\bullet)$ *is* $\mathsf{S}(\mathcal{N}^\bullet)$.

Proof. Since $\mathcal{N}^\bullet$ is concentrated in degree a and $(E\mathcal{N}^\bullet)^n = 0$ for $n < a$ (see proof of 10.1.5), therefore any derived-category isomorphism $\mathcal{N}^\bullet \xrightarrow{\sim} E\mathcal{N}^\bullet$ is represented by a quasi-isomorphism $\mathcal{N}^\bullet \to E\mathcal{N}^\bullet$ which is moreover completely described by the corresponding (injective) map in degree a, $\mathcal{N}^a \to (E\mathcal{N}^\bullet)^a$. Let S be the quasi-isomorphism representing $\mathsf{S}(\mathcal{N}^\bullet)$ and S^a the induced map in degree a. Set $\eta := \eta_f(\mathcal{M}^\bullet)$ and let η^a be the induced map in degree a. The Proposition amounts to saying that $S^a = \eta^a$.

For any sheaf $\mathcal{F}$ on $\mathfrak{X}$ and for any family of $\mathcal{O}_{\mathfrak{X},x}$-modules G_x where x varies over points in $\mathfrak{X}$, if $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X},x}}(\mathcal{F}_x, G_x) = 0$ for all x, then $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \oplus_x i_x G_x) = 0$. In particular, since $(E\mathcal{N}^\bullet)^a$ lies on the Z^a/Z^{a+1}-skeleton, to show that $S^a = \eta^a$, it suffices to show that for any $x \in \mathfrak{X}$ with $\Delta_{\mathfrak{X}}(x) = a$, the natural induced maps S^a_x and η^a_x from $\mathcal{N}^a_x$ to $(E\mathcal{N}^\bullet)^a_x$ are equal. Moreover, by 10.1.3(iii), it suffices to confine our attention to $x \in \mathfrak{X}$ such that $\Delta_{\mathfrak{X}}(x) = a$ and $\Delta_{\mathfrak{Y}}(f(x)) = a + d$. In particular, the identifications of 10.1.3(iv) apply.

It suffices to show that for $\phi = \eta^a_x$ or $\phi = S^a_x$, the following maps compose to the identity map

$$H^a_x\mathcal{N}^\bullet = H^0_x\mathcal{N}^a = \mathcal{N}^a_x \xrightarrow{\phi} (E\mathcal{N}^\bullet)^a_x = (E\mathcal{N}^\bullet)(x) \xrightarrow{(10)} H^a_x\mathcal{N}^\bullet.$$

Indeed, since the remaining maps are isomorphisms, ϕ is uniquely determined by the condition that the composition be identity.

In the case of $\phi = \eta_x^a$ we conclude using the outer border of the following diagram of natural isomorphisms whose top row gives η_x^a and whose commutativity is straightforward to verify.

$$\begin{tikzcd}
\mathcal{N}_x^a \arrow[r] \arrow[d, equal] & M \otimes_y \omega_x \arrow[r] \arrow[d, phantom, "\text{see 10.1.3(iv)}"] & H^0_{m_x}(M \otimes_y \omega_x) \arrow[r, "(-1)^{qd}\times(38)^{-1}"] \arrow[d] & (E\mathcal{N}^\bullet)(x) = (E\mathcal{N}^\bullet)^a_x \arrow[d, "(10)"] \\
\varGamma_x \mathcal{N}^a \arrow[rr, equal] & & H^0_x \mathcal{N}^a \arrow[r, equal] & H^a_x \mathcal{N}^\bullet
\end{tikzcd}$$

In the case of $\phi = S_x^a$ we use the following commutative diagram of natural isomorphisms, where the top row composes to the identity map by 3.3.3.

$$\begin{tikzcd}
H_x^a \mathcal{N}^\bullet \arrow[r, "H_x^a S"] \arrow[d, equal, "H_x^a\, \sigma_{\le a}"] & H_x^a E\mathcal{N}^\bullet \arrow[r, "\text{3.2.1(i)[c]}"] \arrow[d, "H_x^a\, \sigma_{\le a}"] & H_x^a \mathcal{N}^\bullet \arrow[ddd, equal] \\
H_x^a(\mathcal{N}^a[-a]) \arrow[r, "H_x^a(S^a[-a])"] \arrow[d, equal] & H_x^a((E\mathcal{N}^\bullet)^a[-a]) \arrow[d, equal] & \\
\varGamma_x \mathcal{N}^a \arrow[r, "\varGamma_x S^a"] \arrow[d, equal] & \varGamma_x (E\mathcal{N}^\bullet)^a \arrow[d, equal] & \\
\mathcal{N}_x^a \arrow[r, "S_x^a"] & (E\mathcal{N}^\bullet)_x^a \arrow[r, "(10)"] & H_x^a \mathcal{N}^\bullet
\end{tikzcd}$$

□

Now we tackle the general case when $\mathcal{M}^\bullet$ is no longer assumed to be concentrated in one spot.

For integers r, s, t let $\delta^{r,s,t}_{\mathcal{M}^\bullet}$ denote the map $\mathcal{E}^{r,s}_{\mathcal{M}^\bullet} \to \mathcal{E}^{r+1,t}_{\mathcal{M}^\bullet}$ defined as follows.

$$\begin{aligned}
\mathcal{E}^{r,s}_{\mathcal{M}^\bullet} \xrightarrow{\text{inclusion}} \mathcal{E}^{r}_{\mathcal{M}^\bullet} &\xrightarrow{(87)} (Ef^\diamond \mathcal{M}^\bullet)^r \\
&\xrightarrow{\text{differential}} (Ef^\diamond \mathcal{M}^\bullet)^{r+1} \xrightarrow{(87)^{-1}} \mathcal{E}^{r+1}_{\mathcal{M}^\bullet} \xrightarrow{\text{projection}} \mathcal{E}^{r+1,t}_{\mathcal{M}^\bullet}
\end{aligned}$$

LEMMA 10.1.7. *Let* $f, \mathcal{X}, \Delta_{\mathcal{X}}, \mathcal{Y}, \Delta_{\mathcal{Y}}, d,$ *be as in* 10.1.3. *Let* $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$, *no longer assumed to be concentrated in one degree. For any integers* r, s, t *the following hold.*

(i) *If* $s > t$, *then* $\delta^{r,s,t}_{\mathcal{M}^\bullet} = 0$.

(ii) *Assume* $s \le t$. *For any integer* n, *if* $n \le s$ *so that* $\mathcal{E}^{r,s}_{\mathcal{M}^\bullet} = \mathcal{E}^{r,s}_{\sigma_{\ge n}\mathcal{M}^\bullet}$ *and* $\mathcal{E}^{r+1,t}_{\mathcal{M}^\bullet} = \mathcal{E}^{r+1,t}_{\sigma_{\ge n}\mathcal{M}^\bullet}$, *then* $\delta^{r,s,t}_{\mathcal{M}^\bullet} = \delta^{r,s,t}_{\sigma_{\ge n}\mathcal{M}^\bullet}$. *Similarly, if* $n \ge t$, *then* $\delta^{r,s,t}_{\mathcal{M}^\bullet} = \delta^{r,s,t}_{\sigma_{\le n}\mathcal{M}^\bullet}$.

(iii) *For any integer* a, $\delta^{a,a+d,a+d}_{\mathcal{M}^\bullet} = \delta^{a,a+d,a+d}_{\mathcal{M}^{a+d}[-a-d]}$.

PROOF. (i) We claim that more generally, any map from $\mathcal{E}^{r,s}_{\mathcal{M}^\bullet}$ to $\mathcal{E}^{r+1,t}_{\mathcal{M}^\bullet}$ is zero when $s > t$. Indeed, let $i_x G_x$ be a component of $\mathcal{E}^{r,s}_{\mathcal{M}^\bullet}$ and $i_{x'} G_{x'}$ a component of $\mathcal{E}^{r+1,t}_{\mathcal{M}^\bullet}$. It suffices to show that $\mathrm{Hom}_{\mathcal{X}}(i_x G_x, i_{x'} G_{x'}) = 0$ or, more generally, that $x' \notin \overline{\{x\}}$. By definition, $\Delta_{\mathcal{Y}}(f(x)) = s$ and $\Delta_{\mathcal{Y}}(f(x')) = t$. Therefore $s > t \Longrightarrow f(x') \notin \overline{\{f(x)\}} \Longrightarrow x' \notin \overline{\{x\}}$.

(ii) Suppose $n \le s$. The canonical map $\sigma_{\ge n}\mathcal{M}^\bullet \to \mathcal{M}^\bullet$ induces a map of complexes $Ef^\diamond\sigma_{\ge n}\mathcal{M}^\bullet \to Ef^\diamond\mathcal{M}^\bullet$. Consider the following diagrams where the bottom row of the diagram on the left is the same as the top row of the diagram on the right.

$$\begin{tikzcd}
\mathcal{E}^{r,s}_{\sigma_{\ge n}\mathcal{M}^\bullet} \arrow[r, equal] \arrow[d] & \mathcal{E}^{r,s}_{\mathcal{M}^\bullet} \arrow[d] \\
\mathcal{E}^{r}_{\sigma_{\ge n}\mathcal{M}^\bullet} \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\Box_1"] & \mathcal{E}^{r}_{\mathcal{M}^\bullet} \arrow[d, "\wr"'] \\
(Ef^\diamond\sigma_{\ge n}\mathcal{M}^\bullet)^r \arrow[r] \arrow[d] & (Ef^\diamond\mathcal{M}^\bullet)^r \arrow[d] \\
(Ef^\diamond\sigma_{\ge n}\mathcal{M}^\bullet)^{r+1} \arrow[r] & (Ef^\diamond\mathcal{M}^\bullet)^{r+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
(Ef^\diamond\sigma_{\ge n}\mathcal{M}^\bullet)^{r+1} \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\Box_2"] & (Ef^\diamond\mathcal{M}^\bullet)^{r+1} \arrow[d, "\wr"'] \\
\mathcal{E}^{r+1}_{\sigma_{\ge n}\mathcal{M}^\bullet} \arrow[r] \arrow[d] & \mathcal{E}^{r+1}_{\mathcal{M}^\bullet} \arrow[d] \\
\mathcal{E}^{r+1,t}_{\sigma_{\ge n}\mathcal{M}^\bullet} \arrow[r, equal] & \mathcal{E}^{r+1,t}_{\mathcal{M}^\bullet}
\end{tikzcd}$$

The left columns of the two diagrams together give $\delta^{r,s,t}_{\sigma_{\ge n}\mathcal{M}^\bullet}$ while the right ones right give $\delta^{r,s,t}_{\mathcal{M}^\bullet}$. The rectangles $\Box_1, \Box_2$ commute because the isomorphism in (87), which uses (38), factors through truncations. Commutativity of the remaining rectangles is obvious and so the desired conclusion follows. A similar argument works for the corresponding statement when $n \ge t$.

(iii) Using $r = a$ and $s = t = n = a + d$ in (ii) we conclude by truncating on the left and right. $\Box$

Proposition 10.1.8. *Let notation and assumptions be as in 10.1.7. Then the graded map $\eta_f(\mathcal{M}^\bullet)\colon f^\diamond\mathcal{M}^\bullet \to Ef^\diamond\mathcal{M}^\bullet$ defined in* (90) *is a map of complexes.*

Proof. We have to verify that for any integer a, the following diagram commutes.

$$\begin{tikzcd}
(f^\diamond\mathcal{M}^\bullet)^a \arrow[r] \arrow[d] & \mathcal{E}^{a,a+d}_{\mathcal{M}^\bullet} \arrow[r] & \mathcal{E}^{a}_{\mathcal{M}^\bullet} \arrow[r] & (Ef^\diamond\mathcal{M}^\bullet)^a \arrow[d] \\
(f^\diamond\mathcal{M}^\bullet)^{a+1} \arrow[r] & \mathcal{E}^{a+1,a+1+d}_{\mathcal{M}^\bullet} \arrow[r] & \mathcal{E}^{a+1}_{\mathcal{M}^\bullet} \arrow[r] & (Ef^\diamond\mathcal{M}^\bullet)^{a+1}
\end{tikzcd}$$

Let us first verify that for any integer $n \ge 0$, the composite map

$$(f^\diamond\mathcal{M}^\bullet)^a \longrightarrow \mathcal{E}^{a,a+d}_{\mathcal{M}^\bullet} \xrightarrow{\delta^{a,a+d,a+d-n}_{\mathcal{M}^\bullet}} \mathcal{E}^{a+1,a+d-n}_{\mathcal{M}^\bullet}$$

is zero. If $n > 0$ then we conclude by 10.1.7(i). If $n = 0$, then by 10.1.7(iii) we may assume without loss of generality that $\mathcal{M}^\bullet$ is concentrated in degree $a + d$ only and then we conclude by 10.1.5.

We have thus reduced the Proposition to proving that the following diagram commutes.

$$\begin{tikzcd}
(f^\diamond\mathcal{M}^\bullet)^a \arrow[r, "(89)"] \arrow[d] & \mathcal{E}^{a,a+d}_{\mathcal{M}^\bullet} \arrow[d, "\delta^{a,a+d,a+d+1}_{\mathcal{M}^\bullet}"] \\
(f^\diamond\mathcal{M}^\bullet)^{a+1} \arrow[r, "(89)"'] & \mathcal{E}^{a+1,a+1+d}_{\mathcal{M}^\bullet}
\end{tikzcd}$$

By definition, (89) respects the grading on each side indexed by the points of $\mathcal{Y}$. In particular, it suffices to verify that for any immediate specialization $y \rightsquigarrow y'$ in $\mathcal{Y}$ such that $\Delta_{\mathcal{Y}}(y) = a + d$, the following diagram commutes, with the columns being, as above, induced by the differential of $\mathcal{M}^\bullet$ and $Ef^\diamond\mathcal{M}^\bullet$.

$$\begin{array}{ccc} f^\diamond(\mathcal{M}^\bullet, y) & \longrightarrow & \mathcal{E}^a_{\mathcal{M}^\bullet}(y) \\ \big\downarrow & & \big\downarrow \\ f^\diamond(\mathcal{M}^\bullet, y') & \longrightarrow & \mathcal{E}^{a+1}_{\mathcal{M}^\bullet}(y') \end{array}$$

Before simplifying this diagram further, we need some notation. Set $M := \mathcal{M}^\bullet(y)$, $M' := \mathcal{M}^\bullet(y')$. By 10.1.2, the definition of $\mathcal{E}^{a+1}_{\mathcal{M}^\bullet}(y') \subset \mathcal{E}^{a+1,a+1+d}_{\mathcal{M}^\bullet}$ involves a finite direct sum of modules of the type $i_{x'}G_{x'}$ where x' ranges over the set of generic points of the fiber over y' and $G_{x'} = H^0_{m'_x}(M' \otimes_{y'} \omega_{x'})$. Fix one such point x'. Let $x_1, \dots, x_k, \dots, x_l$ be all the points that are generic in the fiber over y where $x_1, \dots, x_k$ are exactly those that specialize to x'. Any such specialization $x_j \rightsquigarrow x'$ is necessarily immediate since $\Delta_{\mathcal{X}}(x_j) = a$ and $\Delta_{\mathcal{X}}(x') = a+1$. Note that the induced map from the summand $\bigoplus_{j>k} i_{x_j}G_{x_j}$ of $\mathcal{E}^a_{\mathcal{M}^\bullet}(y)$ (where $G_{x_j} = H^0_{m_{x_j}}(M \otimes_y \omega_{x_j})$ for any j) to $\mathcal{E}^{a+1}_{\mathcal{M}^\bullet}(y')$ is zero because, $x' \notin \overline{\{x_j\}}$ for $j > k$ implies more generally that $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(i_{x_j}G_{x_j}, i_{x'}G_{x'}) = 0$.

Coming back to the previous diagram, we may restrict the target $\mathcal{E}^{a+1}_{\mathcal{M}^\bullet}(y')$ of the diagram to the component $i_{x'}G_{x'}$ and moreover, since for any sheaf $\mathcal{F}$, we have $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, i_{x'}G_{x'}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{X},x'}}(\mathcal{F}_{x'}, G_{x'})$, we may localize the previous diagram at x'. Then verifying its commutativity reduces to verifying that of the rectangle on the right in the following diagram, where, as expected, α is induced by the differential of $f^\diamond\mathcal{M}^\bullet$ and β is induced by the differential of $Ef^\diamond\mathcal{M}^\bullet$ via (87).

$$\begin{array}{ccccc} M \otimes_{y'} \omega_{x'} & \xrightarrow{\ \sim\ } & (f^\diamond(\mathcal{M}^\bullet, y))_{x'} & \xrightarrow[\text{and then (88)}]{\text{to stalk at } x_j} & \bigoplus_{j \le k} H^0_{m_{x_j}}(M \otimes_y \omega_{x_j}) \\ {\scriptstyle \partial\otimes 1}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & {\scriptstyle \beta}\big\downarrow \\ M' \otimes_{y'} \omega_{x'} & \xrightarrow{\ \sim\ } & (f^\diamond(\mathcal{M}^\bullet, y'))_{x'} & \xrightarrow{(88)} & H^0_{m'_x}(M' \otimes_{y'} \omega_{x'}) \end{array} \tag{91}$$

Let ∂ denote the $\mathcal{O}_{\mathcal{Y},y'}$-linear map $M \to M'$ induced by the differential of $\mathcal{M}^\bullet$. The rectangle on the left clearly commutes. It remains to prove that the outer border of (91) commutes. We apply 7.2.1 in this situation with essentially the same notation as used there. (The map h and its relative dimension n are now denoted by f, d respectively.) Note that in this case we have $p_1 = 0$ and $\mathcal{L} = \omega$. Let $\mathcal{I}$ be the largest coherent ideal defining the closed set $\overline{\{x_1, \dots, x_k\}}$ and let $I = \mathcal{I}_{x'}$ as in 7.2.1. We make the following claims.

(i) The $\mathcal{O}_{\mathcal{X},x'}$-modules $M \otimes_{y'} \omega_{x'}$ and $M' \otimes_{y'} \omega_{x'}$ are I-torsion.
(ii) The top and bottom rows of (91) give the maps μ_1, μ_2 respectively of 7.2.1.
(iii) The map β of (91) equals $(-1)^d\psi$ for ψ as in 7.2.1. (Note that d corresponds to n used in 7.2.1.)

Assuming these claims, (91) commutes by 7.2.1, thus proving the Proposition.

Both (ii) and (iii) are straightforward to verify. The rest of this proof is devoted to showing (i).

Let $\mathcal{K}$ be the largest open coherent ideal in $\mathcal{O}_{\mathcal{Y}}$ defining the closed set $\overline{\{y\}}$. Since f is adic, therefore $\mathcal{J} := \mathcal{K}\mathcal{O}_{\mathcal{X}}$ is an open ideal in $\mathcal{O}_{\mathcal{X}}$. Since the ordinary

scheme $X := (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J})$ is smooth over the ordinary scheme $Y := (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\mathcal{K})$, therefore the generic points of (the irreducible components of) X all map to the generic point of Y, viz., y. In particular, the generic points of the fiber over y correspond exactly to the generic points of X. Note that $\mathcal{J} \subset \mathcal{I}$. Set $\overline{\mathcal{I}} := \mathcal{I}/\mathcal{J}$. Then the closed set defined by $\overline{\mathcal{I}}$ contains $\mathrm{Spec}(\mathcal{O}_{X,x'})$, and hence the stalk $\overline{I}$ of $\overline{\mathcal{I}}$ at x' is a nilpotent ideal. But $\overline{I}$ can be canonically identified with $\mathcal{I}_{x'}/\mathcal{J}_{x'}$. It follows that proving (i) is equivalent to proving that $M \otimes_{y'} \omega_{x'}$ and $M' \otimes_{y'} \omega_{x'}$ are $\mathcal{J}_{x'}$-torsion. Since $\mathcal{J}_{x'} = \mathcal{K}_{y'}\mathcal{O}_{\mathcal{X},x'}$ we therefore reduce to showing that M, M' are $\mathcal{K}_{y'}$-torsion as $\mathcal{O}_{\mathcal{Y},y'}$-modules. In the case of M we conclude from the fact that $\mathcal{K}_{y'}\mathcal{O}_{\mathcal{Y},y} = \mathcal{K}_y = m_y$ and M is an m_y-torsion $\mathcal{O}_{\mathcal{Y},y}$-module. In the case of M' we conclude from the fact that $\mathcal{K}_{y'} = m_{y'}$. □

Proposition 10.1.9. *With notation and assumptions as in* 10.1.8, *the natural map* $\eta_f(\mathcal{M}^\bullet)\colon f^\diamond\mathcal{M}^\bullet \to Ef^\diamond\mathcal{M}^\bullet$ *is a quasi-isomorphism whose derived-category image* $Q\eta_f(\mathcal{M}^\bullet)$ *is* $\mathsf{S}(f^\diamond\mathcal{M}^\bullet)$.

Proof. Set $\eta := \eta_f$. It suffices to verify that $Q\eta(\mathcal{M}^\bullet) = \mathsf{S}(f^\diamond\mathcal{M}^\bullet)$, because that implies that η is a quasi-isomorphism.

By 3.3.3, the validity of $Q\eta(\mathcal{M}^\bullet) = \mathsf{S}(f^\diamond\mathcal{M}^\bullet)$ amounts to that of the punctual statement that, for any $x \in \mathcal{X}$, with $p = \Delta_{\mathcal{X}}(x)$, the composite isomorphism

$$H^p_x f^\diamond\mathcal{M}^\bullet \xrightarrow{H^p_x\eta(\mathcal{M}^\bullet)} H^p_x Ef^\diamond\mathcal{M}^\bullet \xrightarrow[\text{for } \mathcal{F}^\bullet = f^\diamond\mathcal{M}^\bullet]{3.2.1(\mathrm{i})[\mathrm{c}]} H^p_x f^\diamond\mathcal{M}^\bullet$$

is the identity. Let us verify this punctual statement.

Fix $x \in \mathcal{X}$ with $p = \Delta_{\mathcal{X}}(x)$ and $q = \Delta_{\mathcal{Y}}(f(x))$. Since $H^p_x\eta(-)$ and the isomorphism obtained from 3.2.1(i)[c] are functorial in $\mathcal{M}^\bullet$, in view of the natural isomorphisms (3.2.2)

$$H^p_x f^\diamond\mathcal{M}^\bullet \xrightarrow{\sim} H^p_x f^\diamond\sigma_{\le q}\mathcal{M}^\bullet, \qquad H^p_x Ef^\diamond\mathcal{M}^\bullet \xrightarrow{\sim} H^p_x Ef^\diamond\sigma_{\le q}\mathcal{M}^\bullet,$$

we may replace $\mathcal{M}^\bullet$ by $\sigma_{\le q}\mathcal{M}^\bullet$. By a similar argument we may truncate in the other direction and thus we reduce to the case when $\mathcal{M}^\bullet = \mathcal{M}^q[-q]$ is concentrated in only one degree.

By 10.1.6 for $\mathcal{M}^\bullet = \mathcal{M}^q[-q]$, we already have $Q\eta(\mathcal{M}^\bullet) = \mathsf{S}(f^\diamond\mathcal{M}^\bullet)$ and hence the punctual statement corresponding to x now follows by referring back to 3.3.3. □

Remark 10.1.10. Suppose $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ is as in Lemma 10.1.3. Let

$$L_f(\mathcal{M}^\bullet)(= L_f)\colon \mathbb{E}_f(\mathcal{M}^\bullet) = E(f^\diamond\mathcal{M}^\bullet) \xrightarrow{\sim} f^\sharp(\mathcal{M}^\bullet)$$

be the isomorphism in the construction of $f^\sharp(\mathcal{M}^\bullet)$ in § 8.1 (see the second paragraph of *loc.cit.*). We would like to summarize the relationship between the global maps L_f, η_f and various natural local isomorphisms (e.g. (10), (19) or the isomorphisms in Theorem 4.3.1) in a special case, viz. when $\mathcal{M}^\bullet$ is concentrated at a point. The purpose is to gather together the many sign twists we have used into one place.

Suppose x is a point on $\mathcal{X}$. Let $p = \Delta_{\mathcal{X}}(x)$, $y = f(x)$, $q = \Delta_{\mathcal{Y}}(y)$, $S = \widehat{\mathcal{O}_{\mathcal{X},x}}$, $R = \widehat{\mathcal{O}_{\mathcal{Y},y}}$, $\varphi\colon R \to S$ the natural map induced by f, $t = \dim(S/m_RS) = p - q + d$, M a zero dimensional R–module, $\mathcal{F} = i_yM$, $\mathcal{M}^\bullet = \mathcal{F}[-q]$ and set

$$\varepsilon(x) := \Delta_{\mathcal{X}}(x)\Delta_{\mathcal{Y}}(y) + \Delta_{\mathcal{X}}(x).$$

The natural quasi-isomorphism

$$\mathbb{E}_f(\mathcal{M}^\bullet)(x)[-p] = \varGamma_x\,\mathbb{E}_f(\mathcal{M}^\bullet) \longrightarrow \mathbf{R}\varGamma_x\,\mathbb{E}_f\mathcal{M}^\bullet$$

obtained by applying $\varGamma_x$ to an injective resolution $\mathbb{E}_f(\mathcal{M}^\bullet) \to \mathcal{I}^\bullet$ of the flasque complex $\mathbb{E}_f(\mathcal{M}^\bullet)$ gives on applying H^p the isomorphism

$$\rho\colon \mathbb{E}_f(\mathcal{M}^\bullet)(x) \xrightarrow{\sim} H^p_x\mathbb{E}_f(\mathcal{M}^\bullet),$$

which is the inverse of the map in 3.2.1(i)[b]. In view of Propositions 3.3.3 and 10.1.9 and the definitions in (6), (12), (19), (38), (76), we see that the following diagram commutes up to a sign of $(-1)^{\varepsilon(x)}$:

$$\begin{tikzcd}
H^p_x(\mathbb{E}_f(\mathcal{M}^\bullet)) & & \mathbb{E}_f(\mathcal{M}^\bullet)(x) \arrow[ll, "\sim"', "\rho"] \arrow[d, "L(x)"] \\
H^p_x(f^\diamond\mathcal{M}^\bullet) \arrow[u, "H^p_x(Q_{\mathfrak{X}}\eta_f)"] \arrow[d, equal] & & (f^\sharp\mathcal{M}^\bullet)(x) \arrow[d, equal] \\
H^p_x(f^*\mathcal{F}[-q]\otimes\omega_f[d]) \arrow[d, equal] & (-1)^{\varepsilon(x)} & \varphi_\sharp M \arrow[d, "4.3.1{,}I(i)"] \\
H^t_x(f^*\mathcal{F}\otimes\omega_f) \arrow[rr, "(6){,}(19)"'] & & H^t_{m_S}(M\otimes\omega_{S/R})
\end{tikzcd}$$

10.2. Truncations. In this subsection we gather together some properties of $f^\sharp$ with a view to future applications. These results are not used in the next subsection.

Recall that if $\mathcal{F}$ is an abelian sheaf on a topological space X, x a point on X and G an abelian group, then $\mathrm{Hom}_{\mathrm{Sheaves}}(\mathcal{F}, i_xG) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Groups}}(\mathcal{F}_x, G)$. In particular, if $\mathcal{F}_x = 0$, then any map from $\mathcal{F}$ to i_xG is zero.

Lemma 10.2.1. *Let $f : (\mathfrak{X}, \Delta') \to (\mathfrak{Y}, \Delta)$ be a map in $\mathbb{F}_c$ and let $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ be points such that $\Delta'(x) = \Delta(y)$. If x is not closed in its fiber or if $y \ne f(x)$, then for any $\mathcal{O}_{\mathfrak{X},x}$-module C and any $\mathcal{O}_{\mathfrak{Y},y}$-module M,* $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(f_*i_xC, i_yM) = 0$.

Proof. By definition, $\Delta' = f^\sharp\Delta$ and hence $\Delta'(x) \le \Delta(f(x))$ with equality holding only if x is closed in its fiber. Now note that $f_*i_xC = i_{f(x)}C$. Thus if we assume $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(i_{f(x)}C, i_yM) \ne 0$, then $(i_{f(x)}C)_y \ne 0$ and hence $y \in \overline{\{f(x)\}}$. But this implies that $\Delta(f(x)) \le \Delta(y)$ with equality holding only if $y = f(x)$. Since $\Delta'(x) = \Delta(y)$ we therefore obtain that x is closed in its fiber and $y = f(x)$, which is a contradiction. $\square$

Definition 10.2.2. Let $(\mathfrak{Y}, \Delta) \in \mathbb{F}_c$ and let $\mathcal{M}^\bullet \in \mathrm{Coz}_\Delta(\mathfrak{Y})$. For any $\mathcal{O}_{\mathfrak{Y}}$-endomorphism of the total $\mathcal{O}_{\mathfrak{Y}}$-module $T(\mathcal{M}^\bullet)$ of the graded module $\mathcal{M}^\bullet$, we get, for every pair of points y, y' in $\mathfrak{Y}$ an $\mathcal{O}_{\mathfrak{Y}}$-linear map $i_y(\mathcal{M}^\bullet(y)) \to i_{y'}(\mathcal{M}^\bullet(y'))$. We denote by $\delta_{\mathcal{M}^\bullet}(y, y') : i_y(\mathcal{M}^\bullet(y)) \to i_{y'}(\mathcal{M}^\bullet(y'))$ the map induced by the coboundary map on $\mathcal{M}^\bullet$.

Lemma 10.2.3. *Let $f : (\mathfrak{X}, \Delta') \to (\mathfrak{Y}, \Delta)$ be a map in $\mathbb{F}_c$ and let $\mathcal{C}^\bullet$ be a Cousin complex on $(\mathfrak{X}, \Delta')$. Then $\delta_{\mathcal{C}^\bullet}(x, x') = 0$ if $f(x')$ is not a specialization of $f(x)$.*

Proof. If $f(x')$ is not a specialization of $f(x)$ then x' is not a specialization of x and hence any map from $i_x(\mathcal{C}^\bullet(x))$ to $i_{x'}(\mathcal{C}^\bullet(x'))$ is necessarily zero. $\square$

Lemma 10.2.4. *Let $f : (\mathcal{X}, \Delta') \to (\mathcal{Y}, \Delta)$ be a smooth morphism in $\mathbb{F}_c$. Let*

$$0 \longrightarrow \mathcal{F}_1^\bullet \longrightarrow \mathcal{F}_2^\bullet \longrightarrow \mathcal{F}_3^\bullet \longrightarrow 0$$

be an exact sequence of complexes in $\mathrm{Coz}_\Delta(\mathcal{Y})$. Then the induced sequence

$$0 \longrightarrow f^\sharp\mathcal{F}_1^\bullet \longrightarrow f^\sharp\mathcal{F}_2^\bullet \longrightarrow f^\sharp\mathcal{F}_3^\bullet \longrightarrow 0$$

of complexes in $\mathrm{Coz}_{\Delta'}(\mathcal{X})$ is exact.

Proof. Let $x \in \mathcal{X}$. For any integer i set $L_x^i(-) := H_x^i\mathbf{R}\varGamma'_{\mathcal{X}}(f^*(-) \otimes_{\mathcal{X}} \omega_f[d])$. Set $p := \Delta'(x)$. Since f is a flat map and ω_f a flat $\mathcal{O}_{\mathcal{X}}$-module, we get a long exact sequence

$$\dots \longrightarrow L_x^{p-1}(\mathcal{F}_3^\bullet) \longrightarrow L_x^p(\mathcal{F}_1^\bullet) \longrightarrow L_x^p(\mathcal{F}_2^\bullet) \longrightarrow L_x^p(\mathcal{F}_3^\bullet) \longrightarrow L_x^{p+1}(\mathcal{F}_1^\bullet) \longrightarrow \dots$$

By Cohen-Macaulayness (5.1.3) the L_x^i-terms vanish for $i \neq p$ so that we obtain a short exact sequence

$$0 \longrightarrow L_x^p(\mathcal{F}_1^\bullet) \longrightarrow L_x^p(\mathcal{F}_2^\bullet) \longrightarrow L_x^p(\mathcal{F}_3^\bullet) \longrightarrow 0.$$

Since this holds for every $x \in \mathcal{X}$, in view of the isomorphism $L_x^p(-) \xrightarrow{\sim} (f^\sharp(-))(x)$ (see (76), (38)) the desired conclusion follows. $\square$

Let $(\mathcal{X}, \Delta') \xrightarrow{f} (\mathcal{Y}, \Delta)$ be a map in $\mathbb{F}_c$. Then any Cousin complex in $\mathrm{Coz}_{\Delta'}(\mathcal{X})$ admits natural filtrations arising from the codimension function on $\mathcal{Y}$. The filtrations are obtained as follows. For $p, n \in \mathbb{Z}$ set

$$\Sigma_n^p := \{x \in \mathcal{X} \mid \Delta'(x) = n, \Delta(f(x)) \geq p\}$$

and

$$S_p^n := \{x \in \mathcal{X} \mid \Delta'(x) = n, \Delta(f(x)) \leq p\}.$$

Suppose $\mathcal{C}^\bullet \in \mathrm{Coz}_{\Delta'}(\mathcal{X})$. Then $\mathcal{C}^\bullet$ has a natural decreasing filtration $\{F^p(\mathcal{C}^\bullet)\}_p$ by subcomplexes in $\mathrm{Coz}_{\Delta'}(\mathcal{X})$ defined as follows. For $p \in \mathbb{Z}$ define the nth graded piece of $F^p(\mathcal{C}^\bullet)$ by

$$F^p(\mathcal{C}^\bullet)^n := \bigoplus_{x \in \Sigma_n^p} i_x\mathcal{C}^\bullet(x).$$

By 10.2.3, $F^p(\mathcal{C}^\bullet)$ is stable under the action of the coboundary map $\delta_{\mathcal{C}^\bullet}$ on $\mathcal{C}^\bullet$ so that the restriction of $\delta_{\mathcal{C}^\bullet}$ to $F^p(\mathcal{C}^\bullet)$ gives a differential $\delta_{F^p(\mathcal{C}^\bullet)}$ on $F^p(\mathcal{C}^\bullet)$ that makes it a subcomplex of $\mathcal{C}^\bullet$. In a similar vein we can define, for $p \in \mathbb{Z}$, a complex $G_p(\mathcal{C}^\bullet)$ whose nth graded piece is

$$G_p(\mathcal{C}^\bullet)^n := \bigoplus_{x \in S_p^n} i_x\mathcal{C}^\bullet(x).$$

Viewing $G_p(\mathcal{C}^\bullet)$ as the cokernel of the inclusion $F^{p+1}(\mathcal{C}^\bullet) \to \mathcal{C}^\bullet$, we obtain a differential $\delta_{G_p(\mathcal{C}^\bullet)}$ on $G_p(\mathcal{C}^\bullet)$. Note that for any $x, x' \in \mathcal{X}$ if $\Delta(f(x)), \Delta(f(x')) \leq p$, then $\delta_{G_p(\mathcal{C}^\bullet)}(x, x') = \delta_{\mathcal{C}}^\bullet(x, x')$. There results a short exact sequence of Cousin complexes on $(\mathcal{X}, \Delta')$:

$$0 \longrightarrow F^{p+1}(\mathcal{C}^\bullet) \longrightarrow \mathcal{C}^\bullet \longrightarrow G_p(\mathcal{C}^\bullet) \longrightarrow 0. \tag{92}$$

Note that F^p and G_p are functorial on $\mathrm{Coz}_{\Delta'}(\mathcal{X})$.

Let $\mathcal{F}^\bullet \in \mathrm{Coz}_\Delta(\mathscr{Y})$. With $\sigma_{\geq p}$ and $\sigma_{\leq p}$ the truncation functors (§1.4, (x)) we claim that the following hold.

$$
\begin{aligned}
F^p(f^\sharp\mathcal{F}^\bullet) &= f^\sharp(\sigma_{\geq p}\mathcal{F}^\bullet)\\
G_p(f^\sharp\mathcal{F}^\bullet) &= f^\sharp(\sigma_{\leq p}\mathcal{F}^\bullet)\\
F^p(f^\sharp\mathcal{F}^\bullet)/F^{p+1}(f^\sharp\mathcal{F}^\bullet) &= f^\sharp(\mathcal{F}^p[-p]).
\end{aligned}
\tag{93}
$$

All these are straightforward to verify using the arguments in (ii) and (iii) of 10.1.7.

10.3. Étale maps. By an *étale* map of formal schemes we mean a smooth map of relative dimension 0. In the case of an étale map f, the formula for $f^\sharp$ can be simplified further. We begin with a few preliminaries concerning a criterion for a complex to be Cousin.

LEMMA 10.3.1. *Let $(\mathscr{X}, \Delta) \in \mathbb{F}_c$. A complex $\mathcal{F}^\bullet$ on $\mathscr{X}$ is Δ-Cousin if and only if it satisfies the following condition: For any $x \in \mathscr{X}$ and integers i, j such that $i \neq 0$ or $j \neq \Delta(x)$, we have $H^i_x\mathcal{F}^j = 0$.*

PROOF. If $\mathcal{F}^\bullet$ is Δ-Cousin, then the condition stated in the Lemma follows from 3.2.1(i)[a]. Let $Z^\bullet$ be the filtration of $\mathscr{X}$ induced by Δ. For the 'if' part, we must show that for any integer j, $\mathcal{F}^j$ lies on the Z^j/Z^{j+1}-skeleton, i.e.,

$$\mathcal{F}^j = \varGamma_{Z^j}\mathcal{F}^j \qquad \text{and} \qquad H^0_{Z^{j+1}}\mathcal{F}^j = 0 = H^1_{Z^{j+1}}\mathcal{F}^j.$$

Note that the condition of the Lemma can be rephrased as saying that if $i \neq 0$ or $j \neq n$ then $H^i_{Z^n/Z^{n+1}}\mathcal{F}^j = 0$.

Recall that for $p \ll 0$, $Z^p = \mathscr{X}$. Therefore to show that $\mathcal{F}^j = \varGamma_{Z^j}\mathcal{F}^j$, it suffices to show that the natural maps

$$\varGamma_{Z^j}\mathcal{F}^j \to \varGamma_{Z^{j-1}}\mathcal{F}^j \to \varGamma_{Z^{j-2}}\mathcal{F}^j \to \ldots \to \varGamma_{\mathscr{X}}\mathcal{F}^j = \mathcal{F}^j$$

are isomorphisms. For every n there are exact sequences

$$0 \to \varGamma_{Z^{n+1}} \to \varGamma_{Z^n} \to \varGamma_{Z^n/Z^{n+1}} \to 0.$$

By hypothesis, $H^0_{Z^n/Z^{n+1}}\mathcal{F}^j = 0$ for all $j \neq n$. Thus $\varGamma_{Z^{n+1}}\mathcal{F}^j = \varGamma_{Z^n}\mathcal{F}^j$ for $n < j$.

For verifying the other condition of Cousinness, first assume that $\mathscr{X}$ has finite Krull dimension so that Δ is bounded above and hence for $p \gg 0$, $Z^p = \emptyset$. To show that $H^i_{Z^{j+1}}\mathcal{F}^j = 0$ it suffices to show that the natural maps

$$H^i_{Z^{j+1}}\mathcal{F}^j \to H^i_{Z^{j+2}}\mathcal{F}^j \to \ldots \to H^i_{\emptyset}\mathcal{F}^j = 0$$

are isomorphisms. By hypothesis, $H^i_{Z^n/Z^{n+1}}\mathcal{F}^j = 0$ for all i and for $n > j$, whence the desired conclusion follows.

If $\mathscr{X}$ is not finite-dimensional, we use a localization argument, cf. [**11**, p. 242]. For any $x \in \mathscr{X}$ let $\mathscr{X}_{(x)}$ denote the space of all the generizations of x. In fact $\mathscr{X}_{(x)} = \mathrm{Spf}(\widehat{\mathcal{O}_{\mathscr{X},x}})$ where the completion is with respect to the stalk $\mathscr{I}_x$ of any defining ideal $\mathscr{I}$ in $\mathcal{O}_{\mathscr{X}}$. Then $\{Z^p_x := Z^p \cap \mathscr{X}_{(x)}\}_{p\in\mathbb{Z}}$ is the induced filtration on $\mathscr{X}_{(x)}$. Let $f\colon \mathscr{X}_{(x)} \to \mathscr{X}$ be the canonical inclusion. By 10.3.4 below, the condition of the Lemma also holds for the complex $f^{-1}\mathcal{F}^\bullet$ on $\mathscr{X}_{(x)}$. Moreover we have $(H^i_{Z^{j+1}}\mathcal{F}^j)_x \xrightarrow{\sim} (H^i_{Z^{j+1}_x}f^{-1}\mathcal{F}^j)_x$. But the latter is 0 since $\mathscr{X}_{(x)}$ has finite Krull dimension. Since $(H^i_{Z^{j+1}}\mathcal{F}^j)_x = 0$ for every $x \in \mathscr{X}$, therefore $H^i_{Z^{j+1}}\mathcal{F}^j = 0$. $\square$

The next few results concern the localization argument used in the proof of 10.3.1. Lemma 10.3.2 below is used in [**11**, p. 242] without proof.

LEMMA 10.3.2. *Let X be a noetherian topological space such that any nonempty irreducible closed subset Z contains a unique generic point. Let Y be a subset of X that is stable under generization. Let $f\colon Y \to X$ denote the canonical inclusion. Then, for any flasque sheaf $\mathcal{F}$ on X, its restriction $f^{-1}\mathcal{F}$ on Y is also a flasque sheaf.*

PROOF. First note that the *presheaf* restriction of $\mathcal{F}$ is flasque, i.e., if $\mathcal{G}$ denotes the presheaf on Y that assigns to any open subset V of Y, the set $\varinjlim_{U} \mathcal{F}(U)$ where U ranges over open subsets of X containing V, then for any open subset V_1 of Y, the canonical map $\mathcal{G}(Y) \to \mathcal{G}(V_1)$ is surjective. Indeed, if U_1 is any open subset of X containing V_1, then in the following commutative diagram of canonical maps

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\rho_1} & \mathcal{F}(U_1) \\ {\scriptstyle\rho_2}\big\downarrow & & {\scriptstyle\rho_3}\big\downarrow \\ \mathcal{G}(Y) & \xrightarrow{\rho_4} & \mathcal{G}(V_1) \end{array}$$

ρ_1, ρ_2, ρ_3 are all surjective and hence ρ_4 is also surjective. Therefore it suffices to show that $\mathcal{G}$ is already a sheaf. For the rest of this proof the flasqueness assumption on $\mathcal{F}$ is not needed.

By quasi-compactness, it suffices to verify the sheaf property for open coverings consisting of finitely many open sets. Let V be an open subset of Y and let $\{V_i\}_{i=1}^n$ be an open covering of V. Set $V_{ij} := V_i \cap V_j$. Let s_i be sections of $\mathcal{G}$ over V_i such that s_i and s_j agree on V_{ij}. We must show that there is a unique section s of $\mathcal{G}$ over V whose restriction to V_i is s_i. Before proceeding further we remark that for any open subset U of X and any $\alpha \in \mathcal{F}(U)$, if β is the induced element in $\mathcal{G}(U \cap Y)$ then $\operatorname{Supp}(\beta) = \operatorname{Supp}(\alpha) \cap Y$.

By definition, for each i, there exists an open subset U_i of X and an element $t_i \in \mathcal{F}(U_i)$ such that $V_i = U_i \cap V$ and t_i maps to s_i under the canonical map $\mathcal{F}(U_i) \to \mathcal{G}(V_i)$. Set $U_{ij} := U_i \cap U_j$ and set $t_{ij} := t_i - t_j \in \mathcal{F}(U_{ij})$. Since t_{ij} vanishes on $V_{ij} = U_{ij} \cap V$, therefore its support Z_{ij} is a closed subset of U_{ij} avoiding Y. Let $\overline{Z_{ij}}$ denote the closure of Z_{ij} in X. We claim that $\overline{Z_{ij}} \cap Y = \emptyset$.

Since Z_{ij} is an open dense subset of $\overline{Z_{ij}}$, therefore Z_{ij} contains all the generic points of the finitely many irreducible components of $\overline{Z_{ij}}$. Moreover, $\overline{Z_{ij}}$ is the set of all the specializations of these generic points. Since each generic point is outside Y and since Y is stable under generization, it follows that $\overline{Z_{ij}} \cap Y = \emptyset$.

Set $Z := \cup_{i,j} \overline{Z_{ij}}$ and $U'_i := U_i \setminus Z$. Let t'_i be the restriction of t_i to U'_i. Then t'_i and t'_j agree on $U'_i \cap U'_j$ and hence there exists a section t' of $\mathcal{F}$ over $U' := \cup_i U'_i$ whose restriction to U'_i is t'_i. Let s be the image of t' in $\mathcal{G}(V)$. Then the restriction of s to V_i equals s_i. It remains to demonstrate the uniqueness of s.

Let $\hat{s} \in \mathcal{G}(V)$ be such that its restriction to each V_i is s_i. There is an open subset U'' of U' and an element $\hat{t} \in \mathcal{F}(U'')$ such that $\hat{t}$ maps to $\hat{s}$. Let t'' be the restriction of t' to U''. Then the support of $t'' - \hat{t}$ is a closed subset W of U'' that avoids Y. Arguing as above, we see that the closure $\overline{W}$ of W in X also avoids Y. Then the restriction of $\hat{t}$ to the open set $U'' \setminus \overline{W}$ agrees with that of t'' and hence $\hat{s} = s$. $\square$

LEMMA 10.3.3. *Let $f : Y \to X$ be as in 10.3.2. Let $Z \subset X$ be a subset that is stable under specialization. Then for any sheaf $\mathcal{F}$ on X, $f^{-1}\varGamma_Z\mathcal{F} = \varGamma_{Z\cap Y} f^{-1}\mathcal{F}$.*

PROOF. The canonical map $f^{-1}\mathit{\Gamma}_Z\mathcal{F} \to f^{-1}\mathcal{F}$ is injective and factors through $\mathit{\Gamma}_{Z\cap Y} f^{-1}\mathcal{F} \hookrightarrow f^{-1}\mathcal{F}$. Therefore it suffices to check that the natural induced map $\phi\colon f^{-1}\mathit{\Gamma}_Z\mathcal{F} \to \mathit{\Gamma}_{Z\cap Y} f^{-1}\mathcal{F}$ is surjective.

Let V be an open subset of Y. Let $s \in (\mathit{\Gamma}_{Z\cap Y} f^{-1}\mathcal{F})(V)$. Then there exists an open subset U of X and an element $t \in \mathcal{F}(U)$ such that $U \cap Y = V$ and t maps to s under the canonical map $\mathcal{F}(U) \to (f^{-1}\mathcal{F})(V)$. Then $\operatorname{Supp}(t) \cap Y = \operatorname{Supp}(s) \subset Z$. Let W be the set of all specializations of those generic points of (the irreducible components of) $\operatorname{Supp}(t)$ that do not lie in Y. Then W is a closed set avoiding Y. The restriction of t to the open set $U \setminus W$ then has support in Z. Therefore the image of t in $f^{-1}\mathit{\Gamma}_Z\mathcal{F}(V)$ is an element whose image under ϕ is s. □

COROLLARY 10.3.4. *Let X be as in* 10.3.2. *Let x be a point in X and let $Y := X_{(x)}$ be the set of all generizations of x. Let $f\colon Y \to X$ be the canonical inclusion map. Let $Z_2 \subset Z_1$ be subsets of X that are stable under specialization. Set $Z_i' := Z_i \cap Y$. Then for any complex $\mathcal{F}^\bullet$ and any integer i, there is a canonical isomorphism $(H^i_{Z_1'/Z_2'} f^{-1}\mathcal{F}^\bullet)_x \xrightarrow{\sim} (H^i_{Z_1/Z_2}\mathcal{F}^\bullet)_x$.*

PROOF. Since f^{-1} is exact, it preserves quasi-isomorphisms and hence we may replace $\mathcal{F}^\bullet$ by a flasque resolution. By 10.3.2, $f^{-1}\mathcal{F}^\bullet$ consists of flasque sheaves on Y. Thus we reduce to showing that for any flasque sheaf $\mathcal{F}$ on $\mathfrak{X}$, $(H^i\mathit{\Gamma}_{Z_1'/Z_2'} f^{-1}\mathcal{F})_x \xrightarrow{\sim} (H^i\mathit{\Gamma}_{Z_1/Z_2}\mathcal{F})_x$. Since H^i commutes with localization and f^{-1} preserves stalks, we reduce to showing that $\mathit{\Gamma}_{Z_1'/Z_2'} f^{-1}\mathcal{F} = f^{-1}\mathit{\Gamma}_{Z_1/Z_2}\mathcal{F}$. Therefore we conclude from the following commutative diagram where the rows are exact and the two leftmost columns are equalities by 10.3.3.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathit{\Gamma}_{Z_2'} f^{-1}\mathcal{F} & \longrightarrow & \mathit{\Gamma}_{Z_1'} f^{-1}\mathcal{F} & \longrightarrow & \mathit{\Gamma}_{Z_1'/Z_2'} f^{-1}\mathcal{F} & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & f^{-1}\mathit{\Gamma}_{Z_2}\mathcal{F} & \longrightarrow & f^{-1}\mathit{\Gamma}_{Z_1}\mathcal{F} & \longrightarrow & f^{-1}\mathit{\Gamma}_{Z_1/Z_2}\mathcal{F} & \longrightarrow & 0
\end{array}$$

□

PROPOSITION 10.3.5. *Let $f\colon (\mathfrak{X}, \Delta_{\mathfrak{X}}) \to (\mathfrak{Y}, \Delta_{\mathfrak{Y}})$ be a map in $\mathbb{F}_{\mathrm{c}}$ such that for any $x \in \mathfrak{X}$, with $y := f(x)$, $\mathcal{O}_{\mathfrak{X},x}/m_y\mathcal{O}_{\mathfrak{X},x}$ is a finite-dimensional vector space over the residue field $k(y)$ at y (in particular, $\Delta_{\mathfrak{X}}(x) = \Delta_{\mathfrak{Y}}(y)$). Then f has finite fibers. Moreover, for any $\mathcal{M}^\bullet \in \operatorname{Coz}_{\Delta_{\mathfrak{Y}}}(\mathfrak{Y})$, the following hold.*

(i) $\mathit{\Gamma}'_{\mathfrak{X}} f^*\mathcal{M}^\bullet \in \operatorname{Coz}_{\Delta_{\mathfrak{X}}}(\mathfrak{X})$.

(ii) *Let $y \in \mathfrak{Y}$. Set $M := \mathcal{M}^\bullet(y)$. If $f^{-1}\{y\} = \emptyset$, then $\mathit{\Gamma}'_{\mathfrak{X}} f^* i_y M = 0$. Otherwise, if $f^{-1}\{y\} = \{x_1, \dots, x_n\}$, then, with $p = \Delta_{\mathfrak{X}}(x_j) = \Delta_{\mathfrak{Y}}(y)$, the $\Delta_{\mathfrak{X}}$-Cousin complex $\mathit{\Gamma}'_{\mathfrak{X}} f^* i_y M[-p]$ is only concentrated at the points x_j $(1 \le j \le n)$, and for any j there is a natural isomorphism*

$$(\mathit{\Gamma}'_{\mathfrak{X}} f^*\mathcal{M}^\bullet)(x_j) = (\mathit{\Gamma}'_{\mathfrak{X}} f^* i_y M[-p])(x_j) \xrightarrow{\sim} M \otimes_y \mathcal{O}_{\mathfrak{X},x_j}$$

uniquely determined by the natural map

$$\mathit{\Gamma}_{x_j}\mathit{\Gamma}'_{\mathfrak{X}} f^*\mathcal{M}^p = \mathit{\Gamma}_{x_j}\mathit{\Gamma}'_{\mathfrak{X}} f^* i_y M \longrightarrow (f^* i_y M)_{x_j} \xrightarrow{\sim} M \otimes_y \mathcal{O}_{\mathfrak{X},x_j}.$$

PROOF. Let $x \in \mathfrak{X}, y \in \mathfrak{Y}$ with $y = f(x)$. Then the local ring of the fiber space $\mathfrak{X}_y$ at x is isomorphic to $\mathcal{O}_{\mathfrak{X},x}/m_y\mathcal{O}_{\mathfrak{X},x}$ and hence, by hypothesis, has Krull dimension zero. Therefore x does not have any nontrivial generization in $\mathfrak{X}_y$. Since

this holds for any element x in $\mathcal{X}_y$, it follows that $\mathcal{X}_y$ is zero-dimensional and hence consists of finitely many points.

Note that $\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet$ consists of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-modules. For any integer j in view of the canonical decomposition $\mathcal{M}^j \xrightarrow{\sim} \oplus_y\, i_y\mathcal{M}^\bullet(y)$ (where $\Delta_{\mathcal{Y}}(y) = j$), we deduce, using the same arguments as in 3.1.11 that for any $x \in \mathcal{X}$, if $j \neq \Delta_{\mathcal{Y}}(f(x)) = \Delta_{\mathcal{X}}(x)$, then $H^i_x(\varGamma'_{\mathcal{X}} f^*\mathcal{M}^j) = 0$ for all i.

For this paragraph, fix $x \in \mathcal{X}$. Set $y := f(x)$, $p := \Delta_{\mathcal{X}}(x) = \Delta_{\mathcal{Y}}(y)$, $M = \mathcal{M}^\bullet(y)$. Then there are natural isomorphisms

$$H^i_x(\varGamma'_{\mathcal{X}} f^*\mathcal{M}^p) \xrightarrow{\sim} H^i_{m_x}(f^*\mathcal{M}^p)_x \xrightarrow{\sim} H^i_{m_x}(\mathcal{M}^p_y \otimes_y \mathcal{O}_{\mathcal{X},x}) \xrightarrow{\sim} H^i_{m_x}(M \otimes_y \mathcal{O}_{\mathcal{X},x}).$$

By hypothesis, $m_y\mathcal{O}_{\mathcal{X},x}$ is m_x-primary and hence m_y-torsionness of M implies that $M \otimes_y \mathcal{O}_{\mathcal{X},x}$ is m_x-torsion. Thus $H^0_{m_x}(M \otimes_y \mathcal{O}_{\mathcal{X},x}) = M \otimes_y \mathcal{O}_{\mathcal{X},x}$ and moreover $H^i_{m_x}(M \otimes_y \mathcal{O}_{\mathcal{X},x}) = 0$ for $i \neq 0$. In particular, $H^i_x(\varGamma'_{\mathcal{X}} f^*\mathcal{M}^p) = 0$ for $i \neq 0$.

By 10.3.1 it now follows that $\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{X}}}(\mathcal{X})$ and moreover for any $x \in \mathcal{X}$ we obtain a canonical isomorphism $(\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet)(x) \xrightarrow{\sim} \mathcal{M}^\bullet(f(x)) \otimes_{f(x)} \mathcal{O}_{\mathcal{X},x}$.

For (ii), with $y \in \mathcal{Y}$ fixed, we may assume that $\mathcal{M}^\bullet$ is concentrated only at y. It follows from the above description that if $f^{-1}\{y\} = \emptyset$, then for all $x \in \mathcal{X}$, $(\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet)(x) = 0$ and hence $\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet = 0$. Also, if $f^{-1}\{y\} = \{x_1, \dots, x_n\}$ then $\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet$ is only concentrated at the points x_j. By chasing through the isomorphisms in the previous paragraphs for $i = 0$, we obtain the desired isomorphism in (ii). $\square$

Let us note that an étale map satisfies the hypothesis on f in 10.3.5. Indeed, if f is a smooth map of relative dimension d, then d is the sum of the Krull dimension of $(\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x})$ and the transcendence degree of the residue field extension $k(y) \to k(x)$ (cf. 2.6.10). Therefore $d = 0$ implies that $m_y\mathcal{O}_{\mathcal{X},x}$ is m_x-primary and that $k(x)$ is a finite extension of $k(y)$. Thus $(\mathcal{O}_{\mathcal{X},x}/m_y\mathcal{O}_{\mathcal{X},x})$ has finite length over $k(y)$.

We are now in a position to describe $f^\sharp$ for f an étale map. Note that if f is an étale map, then $\omega_f = \mathcal{O}_{\mathcal{X}}$. Moreover, as in the proof of 8.2.1, we shall identify the functor $(-) \otimes \mathcal{O}_{\mathcal{X}}[0]$ with the identity functor.

Proposition 10.3.6. *Let $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ be an étale map in $\mathbb{F}_{\mathrm{c}}$. Let $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Then $\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet$ is canonically isomorphic to $f^\sharp\mathcal{M}^\bullet$. More precisely, with $E = E_{\Delta_{\mathcal{X}}}$, the following canonical maps are isomorphisms*

$$\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \xrightarrow{3.2.1(\mathrm{ii})} E\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \to E\mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \xrightarrow{(76)} f^\sharp\mathcal{M}^\bullet, \tag{94}$$

and for any $x \in \mathcal{X}$, with $y := f(x), M := \mathcal{M}^\bullet(y)$, the corresponding punctual isomorphism is given by

$$(\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet)(x) \xrightarrow{10.3.5(\mathrm{ii})} M \otimes_y \mathcal{O}_{\mathcal{X},x} \xrightarrow{\sim} M \otimes_{\hat{y}} \widehat{\mathcal{O}_{\mathcal{X},x}} \xrightarrow{4.3.1,\ \mathrm{I.(i)}} \widehat{f}_{x\sharp}M. \tag{95}$$

Proof. Let $p = \Delta_{\mathcal{X}}(x)$. We relate (94) with (95) through a diagram, which for convenience, is broken into two parts, viz., (96) and (97) below. The rightmost column of (96) is the same as the leftmost one in (97). The top row of (96) + (97) gives (95) while the leftmost column of (96) gives the first two maps in (94). The remaining portion of the outer border of (96) + (97) gives (76), which is the third map used in (94). Note that in this situation the sign obtained from the map θ occurring in (76) is trivial because now $d = 0$ and $p = q$.

$$
\begin{tikzcd}
\varGamma_{x}(\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{\bullet})^{p} \arrow[rrr, equal] \arrow[d, "\text{3.2.1(ii)}"'] \arrow[drrr, phantom, "\square_1"] & & & \varGamma_{x}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p} \arrow[d, equal] \\
\varGamma_{x}(E\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{\bullet})^{p} \arrow[r] \arrow[d] & H^{p}_{x}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{\bullet} \arrow[r, "\text{truncation}", "\text{on } \mathcal{M}^{\bullet}"'] \arrow[d] & H^{p}_{x}(\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p}[-p]) \arrow[r, equal] \arrow[d] & H^{0}_{x}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p} \arrow[d] \\
\varGamma_{x}(E\mathbf{R}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{\bullet})^{p} \arrow[r] & H^{p}_{x}\mathbf{R}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{\bullet} \arrow[r, "\text{truncation}", "\text{on } \mathcal{M}^{\bullet}"'] & H^{p}_{x}(\mathbf{R}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p}[-p]) \arrow[r, equal] & H^{0}_{x}\mathbf{R}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p}
\end{tikzcd}
\tag{96}
$$

$$
\begin{tikzcd}
\varGamma_{x}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p} \arrow[r] & (\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p})_{x} \arrow[r] & (f^{*}\mathcal{M}^{p})_{x} \arrow[r] & M\otimes_{y}\mathcal{O}_{\mathcal{X},x} \arrow[r] & \widehat{f}_{x\sharp}M \\
H^{0}_{x}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p} \arrow[u, equal] \arrow[r, equal] \arrow[d] & (H^{0}_{\{x\}}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p})_{x} \arrow[u] \arrow[r, "\text{3.1.3}"] \arrow[d] & \varGamma_{m_x}(f^{*}\mathcal{M}^{p})_{x} \arrow[u] \arrow[r] \arrow[d, equal] & \varGamma_{m_x}(M\otimes_{y}\mathcal{O}_{\mathcal{X},x}) \arrow[u] & \\
H^{0}_{x}\mathbf{R}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p} \arrow[r, equal] & (H^{0}_{\{x\}}\mathbf{R}\varGamma'_{\mathcal{X}}f^{*}\mathcal{M}^{p})_{x} \arrow[r, "(6)"] & H^{0}_{m_x}(f^{*}\mathcal{M}^{p})_{x} & &
\end{tikzcd}
\tag{97}
$$

The commutativity of all the subrectangles in (96) and (97) except $\square_1$ are verified easily. For $\square_1$, set $\mathcal{C}^\bullet := \varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet$. We may then rewrite $\square_1$ as the following.

$$\begin{tikzcd}
\varGamma_x \mathcal{C}^p \arrow[rr, equal] \arrow[d] & & \varGamma_x \mathcal{C}^p \arrow[d, equal] \\
\varGamma_x (E\mathcal{C}^\bullet)^p \arrow[r] & H^p_x \mathcal{C}^\bullet \arrow[r] & H^p_x(\mathcal{C}^p[-p])
\end{tikzcd}$$

For commutativity of $\square_1$ we now refer to the proof of 3.2.1(iii). Note that every map in $\square_1$ is an isomorphism.

All the horizontal maps in (96) are isomorphisms and all the maps (horizontal as well as vertical) in (97) are isomorphisms. Therefore the vertical maps in (96), in particular the ones in the leftmost column are also isomorphisms. In particular the middle maps in (94) is also an isomorphism. $\square$

PROPOSITION 10.3.7. *Let $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to (\mathcal{Y}, \Delta_{\mathcal{Y}})$ be an etale map in $\mathbb{F}_{\mathrm{c}}$. Let $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Set $\mathcal{F}^\bullet := \mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet$. Then the following diagram commutes in $\mathbf{D}(\mathcal{X})$ and moreover all the maps in it are $\mathbf{D}(\mathcal{X})$-isomorphisms.*

$$\begin{tikzcd}
\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \arrow[r] \arrow[d, "3.2.1(ii)"] & \mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \arrow[d, "3.3.2"] \\
E\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \arrow[r] & E\mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet
\end{tikzcd}$$

Thus, the Suominen isomorphism $\mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \xrightarrow{\sim} E\mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet$ is the composition of the following (explicit) isomorphisms

$$\mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \xrightarrow{(canonical)^{-1}} \varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \xrightarrow{3.2.1(ii)} E\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \xrightarrow{canonical} E\mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet.$$

PROOF. The vertical maps in the given diagram are $\mathbf{D}(\mathcal{X})$-isomorphisms and by 10.3.6, the bottom row is an isomorphism. Therefore only commutativity of the above diagram needs to be shown. The complexes involved in the above diagram are all $\Delta_{\mathcal{X}}$-CM in $\mathbf{D}(\mathcal{X})$. Let $\overline{\mathbf{C}}, \overline{\mathbf{D}}$ be as in §3.3. Since $E = E_{\Delta_{\mathcal{X}}}$ is fully faithful as a functor from $\overline{\mathbf{D}}$ to $\overline{\mathbf{C}}$ (3.3.1), it suffices to verify that E applied to the above diagram gives a commutative diagram.

Set $\mathcal{C}^\bullet := \varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \in \overline{\mathbf{C}}$ and $\mathcal{N}^\bullet := \mathbf{R}\varGamma'_{\mathcal{X}} f^*\mathcal{M}^\bullet \in \overline{\mathbf{D}}$. There is a canonical map $Q\mathcal{C}^\bullet \to \mathcal{N}^\bullet$ and E of the above diagram may be written as the following one.

$$\begin{tikzcd}
EQ\mathcal{C}^\bullet \arrow[r] \arrow[d, "EQ(\mathcal{C}^\bullet \xrightarrow{\sim} EQ\mathcal{C}^\bullet)"'] & E\mathcal{N}^\bullet \arrow[d, "E(\mathcal{N}^\bullet \xrightarrow{\sim} QE\mathcal{N}^\bullet)"] \\
EQEQ\mathcal{C}^\bullet \arrow[r] & EQE\mathcal{N}^\bullet
\end{tikzcd}$$

The vertical map on the left is also the same as $(EQ\mathcal{C}^\bullet) \xrightarrow{\sim} EQ(EQ\mathcal{C}^\bullet)$ (see proof of 3.3.2) while the vertical map on the right is, by the defining property of the Suominen isomorphism, the same as $(E\mathcal{N}^\bullet) \xrightarrow{\sim} EQ(E\mathcal{N}^\bullet)$. Thus the preceding diagram commutes for functorial reasons. $\square$

Completion of a formal scheme along an open ideal is an etale map. In this case we obtain further concrete descriptions of $(-)^\sharp$. We begin with a preparatory Lemma.

LEMMA 10.3.8. *Let $(\mathcal{Z}, \Delta_z) \in \mathbb{F}_c$. Let $\mathcal{I}$ be an open coherent ideal in $\mathcal{O}_{\mathcal{Z}}$. Let $\kappa\colon (\mathcal{X}, \Delta_x) \to (\mathcal{Z}, \Delta_z)$ be the completion map corresponding to the completion of $\mathcal{Z}$ along $\mathcal{I}$. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{A}_{\mathrm{qct}}(\mathcal{Z})$-modules such that for any integer j, $\mathcal{F}^j$ is a direct sum of modules of the type i_zF_z for $z \in \mathcal{Z}$, where F_z is a zero-dimensional $\mathcal{O}_{\mathcal{Z},z}$-module. Set $\kappa^! := \kappa^{-1}\varGamma_{\mathcal{I}}$. Then the following hold.*

(i) *The canonical maps $\kappa^{-1}\varGamma_{\mathcal{I}}\mathcal{F}^\bullet \to \kappa^*\varGamma_{\mathcal{I}}\mathcal{F}^\bullet \to \varGamma'_{\mathcal{X}}\kappa^*\mathcal{F}^\bullet$ are isomorphisms and for any integer j, $\kappa^{-1}\varGamma_{\mathcal{I}}\mathcal{F}^j$ is a direct sum of modules of the type i_xF_z for $x \in \mathcal{X}$, where $z = \kappa(x)$ and F_z the corresponding $\mathcal{O}_{\mathcal{Z},z}$-module.*
(ii) *Suppose $\mathcal{F}^\bullet \in \mathrm{Coz}_{\Delta_z}(\mathcal{Z})$. Then $\kappa^!\mathcal{F}^\bullet \in \mathrm{Coz}_{\Delta_x}(\mathcal{X})$ and for any $x \in \mathcal{X}$, with $z = \kappa(x), F = \mathcal{F}^\bullet(z)$, we have $(\kappa^!\mathcal{F}^\bullet)(x) = (\kappa^! i_zF)(x) = F$.*
(iii) *Suppose $\mathcal{F}^\bullet$ consists of $\mathcal{A}_{\mathrm{qct}}(\mathcal{Z})$-injectives. Then $\kappa^!\mathcal{F}^\bullet$ consists of $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$-injectives and the canonical map $\varGamma'_{\mathcal{X}}\kappa^*\mathcal{F}^\bullet \to \mathbf{R}\varGamma'_{\mathcal{X}}\kappa^*\mathcal{F}^\bullet$ is an isomorphism.*

PROOF. That $\kappa^{-1}\varGamma_{\mathcal{I}}\mathcal{F}^\bullet \to \kappa^*\varGamma_{\mathcal{I}}\mathcal{F}^\bullet$ is an isomorphism follows immediately from the fact that $\varGamma_{\mathcal{I}}\mathcal{F}^\bullet$ consists of $\mathcal{I}$-torsion modules. At any rate, for (i), since $\kappa^{-1}, \kappa^*, \varGamma_{\mathcal{I}}, \varGamma'_{\mathcal{X}}$ commute with direct sums, we may replace $\mathcal{F}^\bullet$ by a single sheaf $\mathcal{F} := i_zF$. Since $\mathcal{F} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{Z})$ therefore $\varGamma_{\mathcal{I}}\mathcal{F} = \varGamma_{|\mathcal{X}|}\mathcal{F}$ where we identify the underlying topological space $|\mathcal{X}|$ of $\mathcal{X}$ by its image in $\mathcal{Z}$, viz., $\mathrm{Supp}(\mathcal{O}_{\mathcal{Z}}/\mathcal{I})$. If $z \in |\mathcal{X}|$, then $\varGamma_{|\mathcal{X}|}(i_zF) = i_zF$, and if $z \notin |\mathcal{X}|$, then $\varGamma_{|\mathcal{X}|}(i_zF) = 0$. In the former case, if x is the unique point mapping to z, then $\kappa^{-1}i_zF = i_xF$. Moreover, since $\varGamma_{\mathcal{I}}\, i_zF = i_zF$ is an $\mathcal{I}$-torsion module, therefore the natural maps $\kappa^{-1}i_zF \to \kappa^*i_zF \leftarrow \varGamma'_{\mathcal{X}}\kappa^*i_zF$ are isomorphisms. Thus (i) and (ii) follow.

For (iii), by (i) above and 2.3.6, it only remains to be verified that the natural map $\varGamma'_{\mathcal{X}}\kappa^*\mathcal{F}^\bullet \to \mathbf{R}\varGamma'_{\mathcal{X}}\kappa^*\mathcal{F}^\bullet$ is an isomorphism. Since it suffices to prove this locally, we may assume that $\mathcal{I}\mathcal{O}_{\mathcal{X}}$ is generated by global sections. Then we conclude by 3.1.7. $\square$

PROPOSITION 10.3.9. *Let $\kappa, \mathcal{X}, \Delta_x, \mathcal{Z}, \Delta_z$ and $\mathcal{I}$ be as in* 10.3.8. *Then for any $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_z}(\mathcal{Z})$, the canonical graded isomorphism $\kappa^{-1}\varGamma_{\mathcal{I}}\mathcal{M}^\bullet \xrightarrow{\sim} \kappa^\sharp\mathcal{M}^\bullet$, given at the punctual level, with $x \in \mathcal{X}, z = \kappa(x), M = \mathcal{M}^\bullet(z)$, by*

$$(\kappa^{-1}\varGamma_{\mathcal{I}}\mathcal{M}^\bullet)(x) = M \xrightarrow{\sim} M \otimes_z \mathcal{O}_{\mathcal{X},x} \xrightarrow[\text{and 4.3.1, I.(i)}]{\text{completions}} \widehat{\kappa_{x\sharp}}M,$$

is an isomorphism of complexes.

PROOF. It suffices to prove that the outer border of the following diagram commutes

$$\begin{tikzcd}
(\kappa^{-1}\varGamma_{\mathcal{I}}\mathcal{M}^\bullet)(x) \arrow[r, "10.3.8(i)"] \arrow[d, equal] & (\varGamma'_{\mathcal{X}}\kappa^*\mathcal{M}^\bullet)(x) \arrow[r, equal] \arrow[d, "10.3.5(ii)"] & (\varGamma'_{\mathcal{X}}\kappa^*\mathcal{M}^\bullet)(x) \arrow[d, "(95)"] \\
M \arrow[r] & M \otimes_z \mathcal{O}_{\mathcal{X},x} \arrow[r] & \widehat{\kappa_{x\sharp}}M
\end{tikzcd}$$

for then the Proposition follows from 10.3.6 because the isomorphisms in 10.3.8(i) and (94) are maps of complexes. The square on the right commutes by definition of (95). For the square on the left we expand as follows and the required

commutativity is verified easily.

$$\begin{array}{ccccc}
(\kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,\mathcal{M}^\bullet)(x) & \longrightarrow & (\kappa^*\mathit{\Gamma}_{\mathcal{J}}\,\mathcal{M}^\bullet)(x) & \longrightarrow & (\mathit{\Gamma}'_{\mathcal{X}}\kappa^*\mathcal{M}^\bullet)(x) \\
\downarrow & & \downarrow & & \downarrow \\
(\kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,i_zM)_x & \longrightarrow & (\kappa^*\mathit{\Gamma}_{\mathcal{J}}\,i_zM)_x & \longrightarrow & (\mathit{\Gamma}'_{\mathcal{X}}\kappa^*i_zM)_x \\
\downarrow & & \downarrow & & \downarrow \\
(\kappa^{-1}i_zM)_x & \longrightarrow & (\kappa^*i_zM)_x & = & (\kappa^*i_zM)_x \\
\downarrow & & \downarrow & & \downarrow \\
M & \longrightarrow & M\otimes_z\mathcal{O}_{\mathcal{X},x} & = & M\otimes_z\mathcal{O}_{\mathcal{X},x}
\end{array}$$

□

REMARK 10.3.10. In the situation of 10.3.9, for any $z \in \mathcal{Z}$, with $x = \kappa(z)$, the completions of the local rings $\mathcal{O}_{\mathcal{Z},z}$ and $\mathcal{O}_{\mathcal{X},x}$ along their maximal ideals are canonically isomorphic to each other. Therefore, if we make the identification $\widehat{\mathcal{O}_{\mathcal{Z},z}} = \widehat{\mathcal{O}_{\mathcal{X},x}}$, then we find that

$$1^\sharp_{\mathcal{Z}}\kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,\mathcal{M}^\bullet = \kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,1^\sharp_{\mathcal{X}}\mathcal{M}^\bullet = \kappa^\sharp\mathcal{M}^\bullet, \qquad \kappa_*\kappa^\sharp\mathcal{M}^\bullet = \mathit{\Gamma}_{\mathcal{J}}\,1^\sharp_{\mathcal{X}}\mathcal{M}^\bullet.$$

We conclude this subsection with some remarks concerning the Suominen isomorphism S. We have already seen two instances, viz., 10.1.9 and 10.3.7 where an explicit form for S is obtained. In fact we can generalize further. We shall use the principle, already used in the proof of 10.3.7, which is that for any CM complex $\mathcal{N}^\bullet$, to find an explicit candidate for $\mathsf{S}(\mathcal{N}^\bullet)$, it suffices to find an explicit isomorphism from $\mathcal{N}^\bullet$ to a Cousin complex $\mathcal{C}^\bullet$, because, $\mathsf{S}(\mathcal{C}^\bullet)$ is represented by the isomorphism of 3.2.1(ii). We elaborate below.

Let $(\mathcal{X},\Delta_{\mathcal{X}}) \xrightarrow{f_1} (\mathcal{Z},\Delta_{\mathcal{Z}}) \xrightarrow{f_2} (\mathcal{Y},\Delta_{\mathcal{Y}})$ be maps in $\mathbb{F}_{\mathrm{c}}$ where f_1 is an etale map and f_2 is an adic smooth map of relative dimension d. Let $\mathcal{M}^\bullet \in \mathrm{Coz}_{\Delta_{\mathcal{Y}}}(\mathcal{Y})$. Set $f := f_2f_1$. Let us obtain an explicit description of $\mathsf{S}(\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f^\diamond\mathcal{M}^\bullet)$.

There are canonical isomorphisms $f^\diamond\mathcal{M}^\bullet \xrightarrow{\sim} f_1^\diamond f_2^\diamond\mathcal{M}^\bullet = f_1^* f_2^\diamond\mathcal{M}^\bullet$ and hence there are natural explicit $\mathbf{D}(\mathcal{X})$-isomorphisms

$$\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f^\diamond\mathcal{M}^\bullet \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f_1^*f_2^\diamond\mathcal{M}^\bullet \xrightarrow[(90)]{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f_1^*E_{\Delta_{\mathcal{Z}}}f_2^\diamond\mathcal{M}^\bullet \xleftarrow[10.3.7]{\sim} \mathit{\Gamma}'_{\mathcal{X}}f_1^*E_{\Delta_{\mathcal{Z}}}f_2^\diamond\mathcal{M}^\bullet.$$

Since $\mathcal{C}^\bullet := \mathit{\Gamma}'_{\mathcal{X}}f_1^*E_{\Delta_{\mathcal{Z}}}f_2^\diamond$ is a Cousin complex, therefore, via the above isomorphism, $\mathsf{S}(\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f^\diamond\mathcal{M}^\bullet)$ is now given by the isomorphism of 3.2.1(ii).

Now assume further that $f_1 = \kappa$ is the completion map as in 10.3.9. Choose an $\mathcal{A}_{\mathrm{qct}}(\mathcal{Z})$-injective resolution $f_2^\diamond\mathcal{M}^\bullet \to \mathcal{I}^\bullet$. Since κ is flat the induced map $\kappa^*f_2^\diamond\mathcal{M}^\bullet \to \kappa^*\mathcal{I}^\bullet$ is also a quasi-isomorphism. There are natural isomorphisms

$$\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f^\diamond\mathcal{M}^\bullet \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}\kappa^*\mathcal{I}^\bullet \xrightarrow[\text{and } 10.3.8\text{(i)}]{10.3.8\text{(iii)}} \kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,\mathcal{I}^\bullet \xleftarrow{\sim} \kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,E_{\Delta_{\mathcal{Z}}}f_2^\diamond\mathcal{M}^\bullet$$

where the last map is obtained by applying $\kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}$ to the homotopy-unique quasi-isomorphism $Ef_2^\diamond\mathcal{M}^\bullet \to \mathcal{I}^\bullet$ induced by the quasi-isomorphism $\eta_{f_2}(\mathcal{M}^\bullet)$ (10.1.9). Now with $\mathcal{C}^\bullet := \kappa^{-1}\mathit{\Gamma}_{\mathcal{J}}\,E_{\Delta_{\mathcal{Z}}}f_2^\diamond\mathcal{M}^\bullet$, (see 10.3.8(ii)) we see that the isomorphism in 3.2.1(ii) represents $\mathsf{S}(\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}f^\diamond\mathcal{M}^\bullet)$.

The above results can in fact be applied to any smooth map, albeit locally. Indeed, any smooth map $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathbb{F}$ can be factored locally on $\mathcal{X}$ as $f\big|_{\mathcal{U}} = f_2f_1$

where $f_2\colon \mathcal{Z} \to \mathcal{Y}$ is an adic smooth map and $f_1\colon \mathcal{U} \to \mathcal{Z}$ is an étale map. This can be shown by an argument involving locally choosing a differential basis of $\widehat{\Omega}^1_f$. A recent result of Alonso Tarrío, Jeremías López and Pérez Rodríguez [**4**] says that f_1 can in fact be chosen to be a completion map such as κ above. (However their result is stated only for formally smooth maps which are of pseudo-finite type; it is possible though, that their result holds more generally for essentially pseudo-finite type too.) Thus all the descriptions given above may be applied locally.

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[†]Downloadable from <http://www.numdam.org/numdam-bin/feuilleter?j=PMIHES&sl=0>.

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